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NOTE

Discrete-Query Quantum Algorithm for NAND Trees

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Abstract: This is a comment on the article "A Quantum Algorithm for the Hamiltonian NAND Tree" by Edward Farhi, Jeffrey Goldstone, and Sam Gutmann, Theory of Computing 4 (2008) 169–190. That paper gave a quantum algorithm for evaluating NAND trees with running time $O(\sqrt{N})$ in the Hamiltonian query model. In this note, we point out that their algorithm can be converted into an algorithm using $N^{1/2+o(1)}$ queries in the conventional (discrete) quantum query model.

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A NAND tree of depth n is a balanced binary tree whose internal vertices represent NAND gates. Placing bits $x_1,\dots,x_{2^n}$ at the leaves, the root of the NAND tree evaluates to the function $f_n(x_1,\dots,x_{2^n})$, where $f_n:\{0,1\}^{2^n}\to\{0,1\}$ is defined recursively as follows. For $n=0$, $f_0(x)=x$, and for $n>0$,

$$f_n(x_1,\dots,x_{2^n}) = \neg\big(f_{n-1}(x_1,\dots,x_{2^{n-1}}) \wedge f_{n-1}(x_{2^{n-1}+1},\dots,x_{2^n})\big)\,. \tag{1}$$

The goal of the NAND tree problem is to evaluate $f_n(x_1,\dots,x_{2^n})$, making as few queries to the bits $x_1,\dots,x_{2^n}$ as possible. The optimal classical randomized algorithm for this problem makes $\Theta(N^{0.753})$ queries, where $N=2^n$ [9, 10, 11]. Until now, no better quantum algorithm was known, whereas the best known quantum lower bound is only $\Omega(\sqrt{N})$ [2]. Here we show the following.

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Theorem. *The bounded-error quantum query complexity of evaluating balanced binary* NAND *trees is* $N^{1/2+O(1/\sqrt{\log N})}$.

Very recently, Farhi, Goldstone, and Gutmann [6] proposed a quantum algorithm that evaluates NAND trees in time $O(\sqrt{N})$, albeit in the unconventional Hamiltonian oracle model [7, 8] rather than the conventional quantum query model. In their version of the Hamiltonian oracle model, we are given access to a Hamiltonian H_O acting on $n+1$ qubits as

$$H_O|b,k\rangle = -x_k|\neg b,k\rangle \tag{2}$$

for all $b \in \{0,1\}$ and $k \in \{0,1\}^n$, and the goal is to perform the computation using evolution according to $H_O + H_D(t)$ for as short a time as possible, where $H_D(t)$ is an arbitrary driving Hamiltonian (that is possibly time-dependent and may act on an extended Hilbert space).

In the conventional quantum query model, the input is accessible via unitary operations of the form

$$U_O|k,a\rangle = |k,a\oplus x_k\rangle\,, \tag{3}$$

again acting on $n+1$ qubits. Two queries of U_O can be used to implement evolution according to H_O for an arbitrary time t, which can be seen as follows. The procedure acts on states of the form $|b,k,a\rangle$ (where the last register is an ancilla qubit) as follows. First, apply U_O to the second and third registers. Then apply a controlled-$R(t)$ gate with the first register as the target and the third register as the control, where

$$R(t) = \begin{pmatrix} \cos t & i\sin t \\ i\sin t & \cos t \end{pmatrix}. \tag{4}$$

Finally, apply U_O to the second and third registers again. With the ancilla qubit initially in the $|0\rangle$ state, the net effect of this procedure is the mapping $|b,k,0\rangle \mapsto \cos(x_k t)|b,k,0\rangle + i\sin(x_k t)|\neg b,k,0\rangle$, which corresponds to evolution by H_O for time t (that is, the unitary operation $e^{-iH_O t}$).

This simulation of H_O does not imply that any fast algorithm in the Hamiltonian oracle model can be turned into an algorithm with small query complexity in the conventional quantum query model. Accurate simulation of the evolution according to $H_O + H_D(t)$ apparently requires many interleaved evolutions of H_O and $H_D(t)$ each for a small time, yet each of which requires two unitary queries to simulate. Nevertheless, it turns out that a Hamiltonian of the kind used in [6] *can* be simulated in the conventional quantum query model with only small overhead.

Proof of Theorem. In the algorithm of [6], $H_D(t)$ is time-independent, so the evolution for time t is given by $e^{-i(H_O+H_D)t}$. Such evolution according to a sum of time-independent Hamiltonians can be simulated using a high-order approximation of the exponential of a sum in terms of a product of exponentials of the individual terms. As noted in [3, 4], by using a p^{th} order approximation, the simulation can be performed with error at most ε in at most

$$2\frac{5^{2p}(2ht)^{1+1/2p}}{\varepsilon^{1/2p}} \tag{5}$$

steps, where $h = \|H_O + H_D\| \le 3$. This yields a simulation with bounded error in $O(t^{1+1/2p})$ steps for any positive integer p, where the constant implied by the big O notation depends on ε and p. Moreover, setting $p = \sqrt{\log t}$ in Eq. 5, we obtain the bound $t^{1+O(1/\sqrt{\log t})}$ on the number of steps. Since the algorithm of [6] applies H for time $t = O(\sqrt{N})$, the Theorem follows. $\square$

Remark 1. This result can also be deduced by noting that, given query access to the inputs via U_O (Eq. 3), one can easily simulate an oracle for the matrix elements of the underlying Hamiltonian $H_O + H_D$ used in [6], and then applying results in [3, 4] for simulating sparse Hamiltonians.

Remark 2. After the first version of this note appeared [5], the algorithm of [7] was generalized to evaluate an arbitrary AND-OR formula in $N^{1/2+o(1)}$ (discrete) queries [1]. Indeed, by using a discrete-time quantum walk, [1] shows that the bounded-error quantum query complexity of evaluating "approximately balanced" formulas is only $O(\sqrt{N})$. In particular, this improves the above Theorem to show that only $O(\sqrt{N})$ discrete quantum queries suffice to evaluate a balanced binary NAND tree.

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