THEORY OF COMPUTING, Volume 8 (2012), pp. 197–208
www.theoryofcomputing.org

The Communication Complexity of Gap Hamming Distance

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Received: August 8, 2011; published: May 17, 2012.

Abstract: In the *gap Hamming distance* problem, two parties must determine whether their respective strings $x, y \in \{0,1\}^n$ are at Hamming distance less than $n/2 - \sqrt{n}$ or greater than $n/2 + \sqrt{n}$. In a recent tour de force, Chakrabarti and Regev (2010) proved the long-conjectured $\Omega(n)$ lower bound on the randomized communication complexity of this problem. In follow-up work, Vidick (2010) discovered a simpler proof. We contribute a new proof, which is simpler yet and a page-and-a-half long.

1 Introduction

The *gap Hamming distance* problem features two communicating parties, the first of which receives a vector $x \in \{-1,+1\}^n$ and the second a vector $y \in \{-1,+1\}^n$. The two vectors are chosen such that the Hamming distance between them is either noticeably smaller than $n/2$ or noticeably larger than $n/2$. The objective is to reliably determine which is the case by exchanging as few bits of communication as possible. Throughout this paper, communication is assumed to be randomized, and the communication protocol is to produce the correct answer with probability at least $2/3$. Formally, gap Hamming distance is the

*Supported by NSF CAREER award CCF-1149018

ACM Classification: F.1.3

AMS Classification: 68Q17, 68Q25

Key words and phrases: Communication complexity, gap Hamming distance, Talagrand's concentration inequality

 DOI: 10.4086/toc.2012.v008a008

communication problem that corresponds to the following partial function on $\{-1,+1\}^n \times \{-1,+1\}^n$:

$$\mathrm{GHD}_n(x,y) = \begin{cases} -1 & \text{if } \langle x,y\rangle \leq -\sqrt{n}\,, \\ +1 & \text{if } \langle x,y\rangle \geq \sqrt{n}\,, \end{cases} \tag{1.1}$$

where $\langle x,y\rangle = \sum x_i y_i$ is the usual inner product. Note that the function is left undefined when $\langle x,y\rangle$ is small in absolute value. The choice of $\sqrt{n}$ for the gap is natural for reasons of measure and represents the most difficult case from the standpoint of proof. When the gap is different in size or simply shifted to a different location in $\{-n,\ldots,-1,0,1,\ldots,n\}$, one applies a padding argument [7] to reduce the new problem to the above canonical case.

The gap Hamming distance problem, or GHD for short, was proposed by Indyk and Woodruff [8] as a means to understand the complexity of several streaming tasks. More specifically, lower bounds on the communication complexity of gap Hamming distance imply lower bounds on the memory requirements of estimating the number of distinct elements in a data stream. Applications of GHD have been discovered to several other streaming tasks, including the computation of frequency moments [18] and entropy [6].

The communication complexity of gap Hamming distance has been the subject of much study over the past few years. The original paper by Indyk and Woodruff [8] proved a linear lower bound on the *one-way* communication complexity of a problem closely related to GHD. A linear lower bound on the one-way communication complexity of GHD itself was obtained by Woodruff [18], with shorter and more elementary proofs discovered subsequently by Jayram, Kumar, and Sivakumar [10], Woodruff [19], and Brody and Chakrabarti [4]. A few years ago, Brody and Chakrabarti [4] obtained a linear lower bound for *constant-round* protocols solving GHD. The dependence on the number of rounds was improved by Brody et al. [5]. The communication complexity of GHD in the unbounded-round model, on the other hand, proved to be a challenge to analyze. A lower bound of $\Omega(\sqrt{n})$ is immediate by a reduction from the *disjointness* problem [11, 14]. Coincidentally, GHD has a quantum communication protocol with cost $O(\sqrt{n}\log n)$, so that the many quantum methods discovered to date were of no use in moving beyond the $\sqrt{n}$ barrier. Finally, in a recent tour de force, Chakrabarti and Regev [7] settled the problem definitively with a linear lower bound on the unbounded-round communication complexity of gap Hamming distance:

Theorem 1.1 (Chakrabarti and Regev)**.** *If a randomized communication protocol solves* GHD_n *with probability* $2/3$ *on every valid input, then it has communication complexity* $\Omega(n)$.

The proof in [7] is quite involved. In follow-up work, Vidick [17] discovered a simpler proof. This paper contributes a new proof of Theorem 1.1, which is short and simpler than the proofs of Chakrabarti and Regev [7] and Vidick [17]. In what follows, we give a detailed overview of previous work and our approach.

1.1 Some Terminology

Let $f\colon X \times Y \to \{-1,+1\}$ be a given communication problem. A common starting point in proving lower bounds on randomized communication complexity is Yao's minimax theorem [20]: to rule out a randomized protocol for f with cost c and error probability at most ε, one defines a probability distribution μ on $X \times Y$ and argues that with respect to μ, every *deterministic* protocol with cost c errs on more than an ε fraction of the inputs. This approach is *complete* in that one can always prove a tight lower bound on randomized communication in this manner.

The challenge in Yao's program is establishing the hardness of a given distribution μ for deterministic communication protocols. By far the most common solution since the 1980s is the *corruption method*, pioneered by Yao himself [20]. In more detail, a deterministic communication protocol with cost c gives a partition $X \times Y = \bigcup_{i=1}^{2^c} R_i$, where each set R_i is the Cartesian product of a subset of X and a subset of Y. The sets R_i are called *rectangles*, and the output of the deterministic protocol is constant on each rectangle. To prove a lower bound on communication, one defines a probability measure μ on $X \times Y$ and argues that every rectangle R with nontrivial measure is ε-*corrupted* by elements of $f^{-1}(+1)$, in the sense that

$$\mu(R \cap f^{-1}(+1)) > \varepsilon \mu(R \cap f^{-1}(-1)) \tag{1.2}$$

for some constant $\varepsilon > 0$. Provided that $f^{-1}(-1)$ has reasonable measure, (1.2) bounds from below the total number of rectangles in any partition of $X \times Y$. By symmetry, the roles of $+1$ and -1 can be interchanged throughout this argument. Furthermore, the argument applies unchanged to *partial* functions f, whose domain is a proper subset of $X \times Y$.

Over the years, many approaches have been used to prove (1.2). For the uniform distribution, a particularly general method was discovered by Babai, Frankl, and Simon [2] almost thirty years ago. It plays a key role in several subsequent papers and this work. In detail, let μ be the uniform measure on $X \times Y$. For the sake of contradiction, suppose that $R = A \times B$ is a large rectangle that is not ε-corrupt, $A \subseteq X$, $B \subseteq Y$. We may assume that no row of R is 2ε-corrupt because any offending rows can be discarded without affecting the size of R much. The proof is completed in two steps.

STEP 1: IDENTIFYING A HARD CORE.
Using the hypothesis that A is large, one identifies elements $x_1, x_2, \ldots, x_k \in A$ that are "very dissimilar" and collectively "representative" of X. Naturally, what those words mean depends on the context but one gets the right idea by thinking about k random elements of X. Typically k is tiny, exponentially smaller than $|A|$. We will call $\{x_1, x_2, \ldots, x_k\}$ a *hard core* of A because at an intuitive level, these few elements capture the full complexity of A.

STEP 2: CORRUPTION.
Using the hypothesis that B is large and $x_1, x_2, \ldots, x_k$ are representative, one shows that the rectangle $\{x_1, x_2, \ldots, x_k\} \times B$ is 2ε-corrupt, a contradiction to the fact that no row of $R = A \times B$ is 2ε-corrupt.

This program is successful in practice because it is *much* easier to analyze the corruption of a rectangle $\{x_1, x_2, \ldots, x_k\} \times B$ for a small and highly structured collection of elements $x_1, x_2, \ldots, x_k$. Babai, Frankl, and Simon [2] used this approach to establish, with an exceedingly elegant and short proof, an $\Omega(\sqrt{n})$ lower bound on the communication complexity of set disjointness. In that work, X and Y both referred to the family of subsets of $\{1, 2, \ldots, n\}$ of cardinality $\sqrt{n}$, and the hard core used in Step 1 was a collection of $k = \varepsilon\sqrt{n}$ subsets that are mostly disjoint.

1.2 Previous Work

We are now in a position to outline the proofs of Theorem 1.1 due to Chakrabarti and Regev [7] and Vidick [17]. Both works study a continuous version of gap Hamming distance, in which the parties receive inputs $x, y \in \mathbb{R}^n$ drawn according to Gaussian measure and need to determine whether their inner product is less than $-\sqrt{n}$ or greater than $\sqrt{n}$. It was shown earlier [5] that the discrete and continuous versions of gap Hamming distance are equivalent from the standpoint of communication complexity. The proof in [7] has two steps. Let $R = A \times B$ be a rectangle of nonnegligible Gaussian measure. For reasons of measure, we may assume that the vectors in A, B have Euclidean norm $\sqrt{n}$, up to a factor $1 \pm \varepsilon$.

STEP 1: IDENTIFYING A HARD CORE.
Using the hypothesis that A has nontrivial measure, one identifies $\Omega(n)$ vectors $x_1, x_2, \ldots, x_i, \ldots \in A$ that are almost orthogonal. More precisely, the Euclidean norm of the projection of x_i onto $\operatorname{span}\{x_1, \ldots, x_{i-1}\}$ is at most a small constant fraction of the norm of x_i.

In retrospect, a system of near-orthogonal vectors is a natural choice for a hard core because GHD is defined in terms of inner products. *That* such a system of vectors can always be chosen from A was proven by Raz [13], who used this fact to obtain a lower bound for another linear-algebraic communication problem (deciding subspace membership). In the light of the program of Babai, Frankl, and Simon [2], it is tempting to proceed to Step 2 and argue that the rectangle $\{x_1, x_2, \ldots, x_i, \ldots,\} \times B$ is heavily corrupted. Unfortunately, gap Hamming distance *does* have rectangles that are large and almost uncorrupted, and one cannot apply the corruption method directly. Instead, Chakrabarti and Regev [7] prove the following.

STEP 2′: ANTICONCENTRATION.
With probability $\Omega(1)$, a random pair $(x, y) \in \{x_1, x_2, \ldots, x_i, \ldots\} \times B$ has $|\langle x, y\rangle| = \Omega(\sqrt{n})$.

The two steps above immediately give the following statement: for any sets $A, B \subseteq \mathbb{R}^n$ of nonnegligible measure, random vectors $x \in A$, $y \in B$ obey $|\langle x, y\rangle| = \Omega(\sqrt{n})$ with constant probability. This *anticoncentration* result is the technical centerpiece of Chakrabarti and Regev's proof. The authors actually derive a much stronger statement, giving a detailed characterization of the distribution of $\langle x, y\rangle$. To complete the proof, they use a criterion for high communication complexity due to Jain and Klauck [9], known as the *smooth rectangle bound*. Specifically, Chakrabarti and Regev use their anticoncentration result to argue that in any partition of $\mathbb{R}^n \times \mathbb{R}^n$, only a small constant measure of inputs can be covered by large uncorrupted rectangles. Settling this claim requires the introduction of a second measure, call it λ, to account for covering by large rectangles. The smooth rectangle bound [9] was discovered very recently and overcomes limitations of Yao's corruption bound—at the expense of being more challenging to use.

In follow-up work, Vidick [17] discovered a simpler proof of the anticoncentration property for $\langle x, y\rangle$, by taking a *matrix-analytic* view of the problem as opposed to the purely measure- and information-theoretic treatment in [7]. Vidick first shows that for any $A \subseteq \mathbb{R}^n$ of nonnegligible Gaussian measure, the matrix $M = \mathbf{E}_{x \in A}[xx^{\mathsf{T}}]$ has a relatively spread-out spectrum, with a constant fraction of singular values on the order of $\Omega(1)$. Since $\mathbf{E}_{x \in A, y \in B}[\langle x, y\rangle^2] = \mathbf{E}_{y \in B}[y^{\mathsf{T}} M y]$, the author of [17] is able to use this spectral property of M to prove anticoncentration for $\langle x, y\rangle$. Vidick's proof ingeniously exploits the rotation-invariance of Gaussian measure and requires just the Bernstein inequality and the Berry-Esseen theorem for independent Gaussian variables. With anticoncentration established, Vidick uses the Jain-Klauck criterion to prove the lower bound on communication complexity.

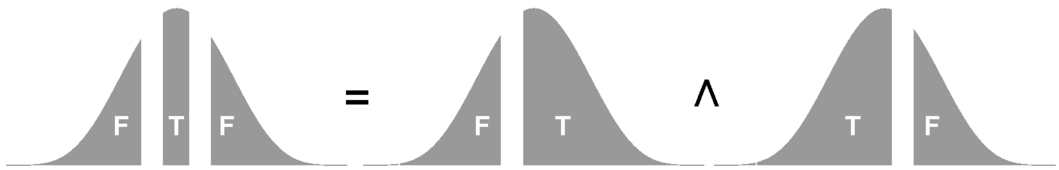

Figure 1: Reduction from gap orthogonality to gap Hamming distance (T = "true," F = "false").

1.3 Our Proof

This paper contributes a new proof of Theorem 1.1. Our approach departs from previous work on two counts. First, we use Yao's original corruption method for proving communication lower bounds, rather than the recent and more involved criterion of Jain and Klauck. Second, the authors of [7, 17] work with an extension of the problem to Gaussian space, whereas we are able to give a direct argument for the hypercube. As we show, the discrete setting allows for a treatment that is much simpler both in formalism and in substance; contrast the proof of Lemma 4.4 in [13] with that of Lemma 3.1 in this paper to get an idea.

Our main technical tool is *Talagrand's concentration inequality* [15, 1, 16]. It states that for any given subset $S \subset \{-1,+1\}^n$ of constant measure, nearly all the points of the hypercube lie at a short Euclidean distance from the convex hull of S. Talagrand's concentration inequality has yielded results whose range and depth are out of proportion to the inequality's easy proof [1, 16]. We use the following well-known consequence of Talagrand's inequality: the projection of a random vector $x \in \{-1,+1\}^n$ onto a given linear subspace $V \subseteq \mathbb{R}^n$ has Euclidean norm $\sqrt{\dim V} \pm O(1)$ almost surely.

We now give a more detailed description of the proof. What we actually obtain is an $\Omega(n)$ lower bound on the communication complexity of *gap orthogonality*, a problem in which the two parties receive vectors $x, y \in \{-1,+1\}^n$ and need to reliably tell whether they are nearly orthogonal or far from orthogonal. Formally, gap orthogonality is the partial Boolean function on $\{-1,+1\}^n \times \{-1,+1\}^n$ given by

$$\mathrm{ORT}_n(x,y) = \begin{cases} -1 & \text{if } |\langle x,y\rangle| \le \sqrt{n}\,, \\ +1 & \text{if } |\langle x,y\rangle| \ge 2\sqrt{n}\,. \end{cases} \tag{1.3}$$

Gap orthogonality readily reduces to gap Hamming distance, as suggested pictorially in Figure 1. Hence, it suffices to prove an $\Omega(n)$ lower bound for gap orthogonality.

It seems at first that nothing of substance is gained by switching from gap Hamming distance to gap orthogonality. In actuality, the latter is preferable in that it allows the use of Yao's corruption method. Indeed, the corruption property for ORT_n is *equivalent* to the anticoncentration of $|\langle x,y\rangle|$. Thus, we just need to establish the anticoncentration: for some absolute constant $\varepsilon > 0$ and any sets $A, B \subseteq \{-1,+1\}^n$ of uniform measure at least $2^{-\varepsilon n}$, the inner product $\langle x,y\rangle$ for random $x \in A$, $y \in B$ cannot be too concentrated around zero. We give a short proof of this result, which combines selected ideas of [7] and [17] with some new elements.

STEP 1: IDENTIFYING A HARD CORE.
One can select a family of $\Omega(n)$ near-orthogonal vectors $x_1, x_2, \dots, x_i, \dots \in A$. Formally, the projection of x_i onto $\operatorname{span}\{x_1, x_2, \dots, x_{i-1}\}$ has Euclidean norm no greater than a *third* of the norm of x_i.

STEP 2: ANTICONCENTRATION & CORRUPTION.
Fix the vectors so constructed. Then with probability exponentially close to 1, a random $y \in \{-1,+1\}^n$ will have nonnegligible inner product (absolute value at least $\sqrt{n}/4$) with one or more of the vectors x_i.

Step 1 is a trivial consequence of Talagrand's concentration inequality; earlier works by Raz [13] and Chakrabarti and Regev [7] used an analogue of this claim for the sphere $\mathbb{S}^{n-1}$ with the uniform measure, whose proof was more involved. To prove Step 2, we switch to the matrix-analytic view of Vidick [17] but give a simpler and more direct argument. Specifically, we consider the matrix M with rows $x_1, x_2, \dots, x_i, \dots$, which by construction is close in norm to an *orthogonal* matrix. It follows that a constant fraction of the singular values of M are large, on the order of $\Omega(\sqrt{n})$. Applying Talagrand a second time, we get that a random vector $y \in \{-1,+1\}^n$ will have a constant fraction of its Euclidean norm in the linear subspace corresponding to the large singular values of M, except with probability exponentially small. This completes Step 2. The desired anticoncentration property falls out as a corollary, for purely combinatorial reasons. This proves corruption for gap orthogonality.

2 Preliminaries

Notation

The symbol $[k]$ stands for the set $\{1,2,\dots,k\}$. The inner product of $x, y \in \mathbb{R}^n$ is denoted $\langle x, y\rangle = \sum x_i y_i$. Likewise, we let $\langle A, B\rangle = \sum A_{ij} B_{ij}$ for matrices $A = [A_{ij}]$ and $B = [B_{ij}]$. The Boolean values "true" and "false" are represented in this paper by -1 and $+1$, respectively. In particular, Boolean functions take on values ± 1. A *partial* function on X is a function whose domain of definition, denoted $\operatorname{dom} f$, is a proper subset of X. For a Boolean string x, the symbol x^k stands for the concatenation $xx\dots x$ (k times).

Linear algebra

The Frobenius norm of a real matrix $M = [M_{ij}]$ is given by $\|M\|_{\mathrm{F}} = (\sum M_{ij}^2)^{1/2}$. We denote the singular values of M by $\sigma_1(M) \geq \sigma_2(M) \geq \cdots \geq 0$. It is straightforward to bound the inner product of matrices A, B in terms of their singular values. Specifically, fixing a singular value decomposition $A = \sum \sigma_i(A) u_i v_i^{\mathsf{T}}$ for some unit vectors u_i, v_i, one arrives at the following well-known bound:

$$\langle A, B\rangle = \sum_i \sigma_i(A) \langle u_i v_i^{\mathsf{T}}, B\rangle = \sum_i \sigma_i(A) u_i^{\mathsf{T}} B v_i \leq \sigma_1(B) \sum_i \sigma_i(A)\,. \tag{2.1}$$

Very precise methods are known for estimating the rth singular value of a matrix, including the *Hoffman-Wielandt inequality*. For us, the following crude bound is all that is needed.

Fact 2.1. *For all real matrices $M, \tilde{M}$ and all natural numbers $r < \operatorname{rk} M$,*

$$\sigma_{r+1}(M) \geq \frac{1}{\operatorname{rk} M - r}\left(\frac{\langle M, \tilde{M}\rangle}{\sigma_1(\tilde{M})} - \|M\|_{\mathrm{F}}\sqrt{r}\right).$$

Proof. Let $\sigma_i = \sigma_i(M)$. The r largest singular values of M sum to $\sigma_1 + \cdots + \sigma_r \leq (\sigma_1^2 + \cdots + \sigma_r^2)^{1/2}\sqrt{r} \leq \|M\|_{\mathrm{F}}\sqrt{r}$, and the remaining ones sum to at most $(\operatorname{rk} M - r)\sigma_{r+1}$. At the same time, $\sum \sigma_i \geq \langle M, \tilde{M}\rangle / \sigma_1(\tilde{M})$ by (2.1). □

The Euclidean norm of a vector $x \in \mathbb{R}^n$ is denoted $\|x\| = (\sum x_i^2)^{1/2}$. The dimension of a linear subspace V is denoted $\dim V$. For a linear subspace $V \subseteq \mathbb{R}^n$ and a vector $x \in \mathbb{R}^n$, we let $\operatorname{proj}_V x$ denote the orthogonal projection of x to V. The following fact is immediate from Talagrand's concentration inequality [1, Thm. 7.6.1].

Fact 2.2 (Talagrand). *For every linear subspace $V \subseteq \mathbb{R}^n$ and every $t > 0$, one has*

$$\mathop{\mathbf{P}}_{x \in \{-1,+1\}^n}\left[\left|\|\operatorname{proj}_V x\| - \sqrt{\dim V}\right| > t\right] < 4\mathrm{e}^{-ct^2},$$

where $c > 0$ is an absolute constant.

Fact 2.2 is entirely classical. The interested reader will find its short derivation in [16] and in Section 5 of this paper.

Communication complexity

Fix finite sets X, Y and let f be a (possibly partial) Boolean function on $X \times Y$. A randomized communication protocol is said to *compute f with error ε* if for all $(x,y) \in \operatorname{dom} f$, the output of the protocol on (x,y) is $f(x,y)$ with probability at least $1 - \varepsilon$. The least communication cost of such a protocol is known as the *ε-error communication complexity of f*, denoted $R_\varepsilon(f)$. For all constants $\varepsilon \in (0, 1/2)$, one has $R_\varepsilon(f) = \Theta(R_{1/3}(f))$.

A *rectangle* of $X \times Y$ is any set of the form $A \times B$, where $A \subseteq X$, $B \subseteq Y$. One of the earliest and best known criteria for high randomized communication complexity is Yao's *corruption bound* [20, 2, 12].

Theorem 2.3 (Corruption bound). *Let f be a (possibly partial) Boolean function on $X \times Y$. Given $\varepsilon, \delta \in (0,1)$, suppose that there is a distribution μ on $X \times Y$ such that*

$$\mu(R \cap f^{-1}(+1)) > \varepsilon\mu(R \cap f^{-1}(-1))$$

for every rectangle $R \subseteq X \times Y$ with $\mu(R) > \delta$. Then

$$2^{R_\xi(f)} \geq \frac{1}{\delta}\left(\mu(f^{-1}(-1)) - \frac{\xi}{\varepsilon}\right).$$

We gave an informal proof of Theorem 2.3 in the Introduction. See [3, Lem. 3.5] for a rigorous treatment.

3 Corruption of Gap Orthogonality

We start by showing that any subset of $\{-1,+1\}^n$ of nontrivial size contains $n/10$ near-orthogonal vectors.

Lemma 3.1. *Let $\alpha > 0$ be a sufficiently small constant. Fix $A \subseteq \{-1,+1\}^n$ with $|A| > 2^{(1-\alpha)n}$. Then for $k = \lfloor n/10 \rfloor$ there exist $x_1, x_2, \dots, x_k \in A$ such that for each i,*

$$\left\| \operatorname{proj}_{\operatorname{span}\{x_1,x_2,\dots,x_i\}} x_{i+1} \right\| \leq \frac{\sqrt{n}}{3}. \tag{3.1}$$

Proof. The proof is by induction. Having selected $x_1, x_2, \dots, x_i \in A$, pick $x_{i+1} \in \{-1,+1\}^n$ uniformly at random. Then $\mathbf{P}[x_{i+1} \in A] > 2^{-\alpha n}$. On the other hand, Fact 2.2 implies that the projection of x_{i+1} to $\operatorname{span}\{x_1, x_2, \dots, x_i\}$ has Euclidean norm at most $\sqrt{n}/3$, except with probability $2^{-\alpha n}$. Thus, there exists $x_{i+1} \in A$ that obeys (3.1). □

Next, we show that given any family of near-orthogonal vectors in $\{-1,+1\}^n$, a random vector in $\{-1,+1\}^n$ almost surely has a substantial inner product with some vector from the family. The proof uses the *lower* bound in Talagrand's theorem, as opposed to Lemma 3.1 in which we used the *upper* bound.

Lemma 3.2. *Fix vectors $x_1, x_2, \dots, x_m \in \{-1,+1\}^n$ that obey (3.1) for all i. Then*

$$\mathop{\mathbf{P}}_{y\in\{-1,+1\}^n} \left[\max_{i=1,\dots,m} |\langle y, x_i \rangle| \leq \frac{\sqrt{n}}{4} \right] \leq \mathrm{e}^{-\beta m} \tag{3.2}$$

for some absolute constant $\beta > 0$.

Proof. Let $\tilde{x}_1, \dots, \tilde{x}_m$ be orthogonal vectors obtained from $x_1, \dots, x_m$ by the Gram-Schmidt process, i. e., $\tilde{x}_i = x_i - \operatorname{proj}_{\operatorname{span}\{x_1,\dots,x_{i-1}\}} x_i$. Let M and $\tilde{M}$ be the $m \times n$ matrices with rows $x_1, \dots, x_m$ and $\tilde{x}_1, \dots, \tilde{x}_m$, respectively. Then

$$\langle M, \tilde{M} \rangle = \sum_{i=1}^m \langle x_i, \tilde{x}_i \rangle = \sum_{i=1}^m \|x_i\|^2 - \sum_{i=1}^m \left\| \operatorname{proj}_{\operatorname{span}\{x_1,\dots,x_{i-1}\}} x_i \right\|^2 \geq \frac{8}{9} nm$$

by (3.1). Also $\sigma_1(\tilde{M}) \leq \sqrt{n}$ since $\tilde{M}$ has orthogonal rows. Thus, Fact 2.1 gives $\sigma_{\lceil m/4 \rceil}(M) \geq 0.51\sqrt{n}$.

Let $M = \sum_{i=1}^m \sigma_i(M) u_i v_i^{\mathsf{T}}$ be the singular value decomposition of M. Define $V = \{v_i : \sigma_i(M) \geq 0.51\sqrt{n}\}$, so that $|V| \geq m/4$ by the previous paragraph. For all vectors y,

$$\|My\|^2 = \sum_{i=1}^m \sigma_i(M)^2 \langle y, v_i \rangle^2 \geq 0.26n \sum_{v \in V} \langle y, v \rangle^2 = 0.26n \left\| \operatorname{proj}_{\operatorname{span} V} y \right\|^2.$$

Since $\operatorname{span} V$ has dimension at least $m/4$, Fact 2.2 guarantees that $\|My\|^2 > mn/16$ with probability $1 - \exp(-\Omega(m))$ for random $y \in \{-1,+1\}^n$. This implies (3.2). □

The main result of this section is immediate from the previous two lemmas for basic combinatorial reasons; cf. the well-known proof of an $\Omega(\sqrt{n})$ lower bound on the communication complexity of disjointness due to Babai, Frankl, and Simon [2].

Theorem 3.3. *Let $\varepsilon > 0$ be a small enough constant. Let $A, B \subseteq \{-1,+1\}^n$ be given such that $\mathbf{P}_{x\in A, y\in B}[x \curlywedge y] \geq 1-\varepsilon$, where $x \curlywedge y$ is shorthand for $|\langle x,y\rangle| \leq \sqrt{n}/4$ ("almost orthogonal"). Then*

$$4^{-n}|A|\,|B| = \exp(-\Omega(n)).$$

Proof. Assume that $|A| > 2\cdot 2^{(1-\alpha)n}$, where $\alpha > 0$ is the constant from Lemma 3.1. We will show that a random $y \in \{-1,+1\}^n$ occurs in B with probability at most $\exp(-\Omega(n))$.

The argument is a combinatorial accounting for what kinds of elements arise in B and is closely analogous to Theorem 8.3 in [2]. Define $A' = \{x \in A : \mathbf{P}_{y\in B}[x \curlywedge y] \geq 1-2\varepsilon\}$, so that $|A'| \geq \frac{1}{2}|A|$. Fix $x_1, x_2, \ldots, x_k \in A'$ to obey (3.1) for all i, where $k = \lfloor n/10 \rfloor$. Define $B' = \{y \in B : \mathbf{P}_{i\in[k]}[x_i \curlywedge y] \geq 1-3\varepsilon\}$, so that $|B'| \geq \frac{1}{3}|B|$. Then $2^{-n}|B'|$ is a lower bound on the probability that a random $y \in \{-1,+1\}^n$ has $|\langle y, x_i\rangle| \leq \sqrt{n}/4$ for at least $(1-3\varepsilon)k$ indices i. By Lemma 3.2 and the union bound, this probability cannot exceed $\binom{k}{3\varepsilon k}\mathrm{e}^{-\Omega(k)} \leq \mathrm{e}^{-\Omega(n)}$. ☐

4 Main Result

In this final section, we prove the sought $\Omega(n)$ lower bound on the communication complexity of gap Hamming distance and gap orthogonality, defined in (1.1) and (1.3).

Main Theorem. $R_{1/3}(\mathrm{ORT}_n) = \Omega(n)$.

Proof. Consider the partial Boolean function f_n on $\{-1,+1\}^n \times \{-1,+1\}^n$ defined as -1 when $|\langle x,y\rangle| \leq \sqrt{n}/8$ and $+1$ when $|\langle x,y\rangle| \geq \sqrt{n}/4$. With μ the uniform distribution on $\{-1,+1\}^n \times \{-1,+1\}^n$, Theorem 3.3 guarantees that $\mu(R \cap f_n^{-1}(+1)) > \varepsilon\mu(R)$ for all rectangles R with $\mu(R) > 2^{-\varepsilon n}$, where $\varepsilon > 0$ is a small constant. Since $\mu(f_n{}^{-1}(-1)) = \Theta(1)$, the corruption bound (Theorem 2.3) shows that $R_{1/3}(f_n) = \Omega(n)$. Finally, $f_n(x,y) = \mathrm{ORT}_{64n}(x^{64}, y^{64})$. ☐

Corollary. $R_{1/3}(\mathrm{GHD}_n) = \Omega(n)$.

Proof. Immediate from the reduction in Figure 1. Formally, for n a square,

$$\begin{aligned}\mathrm{ORT}_n(x,y) = \mathrm{GHD}_{10n+15\sqrt{n}}&\left(x^{10}(-1)^{15\sqrt{n}}, y^{10}(+1)^{15\sqrt{n}}\right)\\ &\wedge \neg\mathrm{GHD}_{10n+15\sqrt{n}}\left(x^{10}(+1)^{15\sqrt{n}}, y^{10}(+1)^{15\sqrt{n}}\right).\end{aligned}$$ ☐

5 More on Talagrand's Concentration Inequality

In this concluding section, we provide additional background on Talagrand's concentration inequality for the interested reader and include the well-known derivation of Fact 2.2. For a nonempty subset $S \subseteq \mathbb{R}^n$ and a point $x \in \mathbb{R}^n$, let $\rho(x,S) = \inf_{y\in S}\|x-y\|$ be the Euclidean distance from x to S. Talagrand's inequality [15, 1] states that for any reasonably large subset S of the hypercube, almost all the points of the hypercube lie at a short distance from the convex hull of S. In more detail:

Theorem 5.1 (Talagrand)**.** *For a fixed convex set $S \subseteq \mathbb{R}^n$ and a random $x \in \{-1,+1\}^n$,*

$$\mathbf{P}[x \in S]\,\mathbf{P}[\rho(x,S) > t] \le \mathrm{e}^{-t^2/16}.$$

We now explain how Fact 2.2 is a consequence of Talagrand's concentration inequality. For any linear subspace $V \subseteq \mathbb{R}^n$ and vector $x \in \mathbb{R}^n$, elementary linear algebra gives $\|\operatorname{proj}_V x\| = \rho(x, V^\perp)$, where $V^\perp$ is the orthogonal complement of V with $\dim V^\perp = n - \dim V$. This allows one to reformulate Fact 2.2 in the language of Euclidean distances: for a fixed linear subspace $V \subseteq \mathbb{R}^n$ and a random $x \in \{-1,+1\}^n$,

$$\mathbf{P}\left[\left|\rho(x,V) - \sqrt{n - \dim V}\right| > t\right] \le 4\mathrm{e}^{-\Omega(t^2)}. \tag{5.1}$$

The advantage of this reformulation is that it can be easily proved using Talagrand's concentration inequality. We present here the proof from a recent expository article by Tao [16]. Fix $a \ge 0$ and consider the convex set $S = \{v \in \mathbb{R}^n : \rho(v,V) \le a\}$. Then $\mathbf{P}[\rho(x,V) \le a]\,\mathbf{P}[\rho(x,S) > t] \le \exp(-t^2/16)$ by Talagrand. But by the triangle inequality, $\rho(x,V) > a + t$ implies $\rho(x,S) > t$, whence

$$\mathbf{P}[\rho(x,V) \le a]\,\mathbf{P}[\rho(x,V) > a+t] \le \mathrm{e}^{-t^2/16} \tag{5.2}$$

for any a. Let m be the median value of $\rho(x,V)$. The two tail bounds

$$\mathbf{P}[\rho(x,V) > m+t] \le 2\mathrm{e}^{-t^2/16}, \tag{5.3}$$

$$\mathbf{P}[\rho(x,V) \le m-t] \le 2\mathrm{e}^{-t^2/16} \tag{5.4}$$

result from letting $a = m$ and $a = m - t$, respectively, in (5.2). Consequently, $\rho(x,V)$ is sharply concentrated around its median. The sharp concentration means among other things that the median is within an additive constant of $\mathbf{E}[\rho(x,V)^2]^{1/2} = \sqrt{n - \dim V}$. Along with (5.3) and (5.4), this settles (5.1).

Acknowledgments

The author is thankful to Amit Chakrabarti, Dmitry Gavinsky, Oded Regev, Thomas Vidick, Ronald de Wolf, David Woodruff, and the anonymous reviewers for their valuable feedback on an earlier version of this manuscript.

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