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Computing the Partition Function for Cliques in a Graph

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Abstract: We present a deterministic algorithm which, given a graph G with n vertices and an integer $1 < m \leq n$, computes in $n^{O(\ln m)}$ time the sum of weights $w(S)$ over all m-subsets S of the set of vertices of G, where $w(S) = \exp\{\gamma m \sigma(S) + O(1/m)\}$ provided exactly $\binom{m}{2}\sigma(S)$ pairs of vertices of S span an edge of G for some $0 \leq \sigma(S) \leq 1$, and $\gamma > 0$ is an absolute constant. This allows us to tell apart the graphs that do not have m-subsets of high density from the graphs that have sufficiently many m-subsets of high density, even when the probability to hit such a subset at random is exponentially small in m.

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1 Introduction and main results

1.1 Density of a subset in a graph

Let $G = (V, E)$ be an undirected graph with set V of vertices and set E of edges, without loops or multiple edges. Let $S \subset V$ be a subset of the set of vertices of G. We define the *density* $\sigma(S)$ of S as the ratio of the number of the edges of G with both endpoints in S to the maximum possible number of edges spanned by vertices in S:

$$\sigma(S) = \frac{|\{u, v\} \in E : u, v \in S|}{\binom{|S|}{2}}.$$

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Hence

$$0 \leq \sigma(S) \leq 1 \quad \text{for all} \quad S \subset V.$$

In particular, $\sigma(S) = 0$ if and only if S is an *independent set*, that is, no two vertices of S span an edge of G and $\sigma(S) = 1$ if and only if S is a *clique*, that is, every two vertices of S span an edge of G.

In this paper, we suggest a new approach to testing the existence of subsets S of a given size $m = |S|$ with high density $\sigma(S)$. Namely, we present a deterministic algorithm, which, given a graph G with n vertices and an integer $1 < m \leq n$ computes in $n^{O(\ln m)}$ time (within a relative error of 0.1, say) the sum

$$\text{Density}_m(G) = \sum_{\substack{S \subset V \\ |S|=m}} \exp\{\gamma m \sigma(S) - \varepsilon(m)\} \quad \text{where} \quad 0 \leq \varepsilon(m) \leq \frac{0.1}{m-1} \tag{1.1}$$

and $\gamma > 0$ is an absolute constant: we can choose $\gamma = 0.07$ and if $n \geq 8m$ and $m \geq 10$ then we can choose $\gamma = 0.27$.

Let us fix two real numbers $0 \leq \sigma < \sigma' \leq 1$. If there are no m-subsets S with density σ or higher then

$$\text{Density}_m(G) \leq \binom{n}{m} \exp\{\gamma m \sigma - \varepsilon(m)\}.$$

If, however, there are sufficiently many m-subsets S with density σ' or higher, that is, if the probability to hit such a subset at random is at least $2\exp\{-\gamma(\sigma' - \sigma)m\}$, then

$$\text{Density}_m(G) \geq 2\binom{n}{m} \exp\{\gamma m \sigma - \varepsilon(m)\}$$

and we can distinguish between these two cases in $n^{O(\ln m)}$ time. Note that "many subsets" still allows for the probability to hit such a subset at random to be exponentially small in m.

It turns out that we can compute in $n^{O(\ln m)}$ time an m-subset S, which is almost as dense as the average under the exponential weighting of the formula (1.1), that is, an m-subset S satisfying

$$\exp\{\gamma m \sigma(S)\} \geq \frac{1}{2}\binom{n}{m}^{-1} \text{Density}_m(G), \tag{1.2}$$

cf. Remark 3.7.

We compute the expression (1.1) using partition functions.

1.2 The partition function of cliques

Let $W = (w_{ij})$ be a set of real or complex numbers indexed by unordered pairs $\{i, j\}$ where $1 \leq i \neq j \leq n$ and interpreted as a set of weights on the edges of the complete graph with vertices $1, \ldots, n$. We write w_{ij} instead of $w_{\{i,j\}}$ and often refer to W as an $n \times n$ symmetric matrix, which should not lead to a confusion as we never attempt to access the diagonal entries w_{ii}.

For an integer $1 < m \leq n$, we define a polynomial

$$P_m(W) = \sum_{\substack{S \subset \{1,\ldots,n\} \\ |S|=m}} \prod_{\substack{\{i,j\} \subset S \\ i \neq j}} w_{ij}\,,$$

which we call the *partition function* of cliques (note that it is different from what is known as the partition function of independent sets, see [20] and [2]). Thus if W is the adjacency matrix of a graph G with set $V = \{1,\ldots,n\}$ of vertices and set E of edges, so that

$$w_{ij} = \begin{cases} 1 & \text{if } \{i,j\} \in E, \\ 0 & \text{otherwise,} \end{cases}$$

then $P_m(W)$ is the number of cliques in G of size m. Hence computing $P_m(W)$ is at least as hard as counting cliques. Let us choose an $0 < \alpha < 1$ and modify the weights W as follows:

$$w_{ij} = \begin{cases} 1 & \text{if } \{i,j\} \in E, \\ \alpha & \text{otherwise.} \end{cases}$$

Then $P_m(W)$ counts all subsets of m vertices in the complete graph on n vertices, only that a subset spanning exactly k non-edges of G is weighted down by a factor of α^k. In other words, an m-subset $S \subset V$ is counted with weight

$$\exp\left\{ \big(1-\sigma(S)\big) \binom{m}{2} \ln\alpha \right\}.$$

In this paper, we present a deterministic algorithm, which, given an $n \times n$ symmetric matrix $W = (w_j)$ such that

$$\big|w_{ij} - 1\big| \;\leq\; \frac{\gamma}{m-1} \quad \text{for all} \quad 1 \leq i \neq j \leq n \tag{1.3}$$

computes the value of $P_m(W)$ within relative error ε in $n^{O(\ln m - \ln\varepsilon)}$ time. Here $\gamma > 0$ is an absolute constant; we can choose $\gamma = 0.07$ and if $n \geq 8m$ and $m \geq 10$, we can choose $\gamma = 0.27$.

1.3 The idea of the algorithm

Let J denote the $n \times n$ matrix filled with 1s. Given an $n \times n$ symmetric matrix W, we consider the univariate function

$$f(t) = \ln P_m\big(J + t(W-J)\big). \tag{1.4}$$

Clearly,

$$f(0) = \ln P_m(J) = \ln\binom{n}{m} \quad \text{and} \quad f(1) = \ln P_m(W)\,.$$

Hence our goal is to approximate $f(1)$ and we do it by using the Taylor polynomial expansion of f at $t = 0$:

$$f(1) \approx f(0) + \sum_{k=1}^{\ell} \frac{1}{k!} \frac{d^k}{dt^k} f(t)\Big|_{t=0}. \tag{1.5}$$

It turns out that the right hand side of the approximation (1.5) can be computed in $n^{O(\ell)}$ time. We present the algorithm in Section 2. The quality of the approximation (1.5) depends on the location of *complex* zeros of P_m.

Lemma 1.1. *Suppose that there exists a real $\beta > 1$ such that*

$$P_m\big(J + z(W-J)\big) \neq 0 \quad \textit{for all} \quad z \in \mathbb{C} \quad \textit{satisfying} \quad |z| \leq \beta.$$

Then the right hand side of the formula (1.5) approximates $f(1)$ within an additive error of

$$\frac{m(m-1)}{2(\ell+1)\beta^{\ell}(\beta-1)}.$$

In particular, for a fixed $\beta > 1$, to ensure an additive error of $0 < \varepsilon < 1$, we can choose $\ell = O(\ln m - \ln \varepsilon)$, which results in the algorithm for approximating $P_m(W)$ within relative error ε in $n^{O(\ln m - \ln \varepsilon)}$ time. We prove Lemma 1.1 in Section 2.

Thus we have to identify a class of weights W for which the number $\beta > 1$ of Lemma 1.1 exists. We prove the following result.

Theorem 1.2. *There is an absolute constant $\omega > 0$ (one can choose $\omega = 0.071$, and if $m \geq 10$ and $n \geq 8m$, one can choose $\omega = 0.271$) such that for any positive integers $1 < m \leq n$, for any $n \times n$ symmetric complex matrix $Z = (z_{ij})$, we have*

$$P_m(Z) \neq 0$$

as long as

$$\left|z_{ij} - 1\right| \leq \frac{\omega}{m-1} \quad \textit{for all} \quad 1 \leq i \neq j \leq n.$$

We prove Theorem 1.2 in Section 3. Theorem 1.2 implies that if the inequality (1.3) holds, where $0 < \gamma < \omega$ is an absolute constant, we can choose $\beta = \omega/\gamma$ in Lemma 1.1 and obtain an algorithm which computes $P_m(W)$ within a given relative error ε in $n^{O(\ln m - \ln \varepsilon)}$ time. Thus we can choose $\gamma = 0.07$ and $\beta = 71/70$ and if $m \geq 10$ and $n \geq 8m$, we can choose $\gamma = 0.27$ and $\beta = 271/270$.

1.4 Weighted enumeration of subsets

Given a graph G with set of vertices $V = \{1, \ldots, n\}$ and set E of edges, we define weights W by

$$w_{ij} = \begin{cases} 1 + \frac{\gamma}{m-1} & \text{if } \{i, j\} \in E, \\ 1 - \frac{\gamma}{m-1} & \text{otherwise,} \end{cases} \tag{1.6}$$

where $\gamma > 0$ is the absolute constant in the inequality (1.3). Thus we can choose $\gamma = 0.07$ and if $m \geq 10$ and $n \geq 8m$, we can choose $\gamma = 0.27$. We define the functional (1.1) by

$$\text{Density}_m(G) = e^{\gamma m}\left(1 + \frac{\gamma}{m-1}\right)^{-\binom{m}{2}} P_m(W).$$

Then we have

$$\mathrm{Density}_m(G) = \sum_{\substack{S \subset V \\ |S|=m}} w(S),$$

where

$$\begin{aligned} w(S) &= e^{\gamma m}\left(1+\frac{\gamma}{m-1}\right)^{(\sigma(S)-1)\binom{m}{2}}\left(1-\frac{\gamma}{m-1}\right)^{(1-\sigma(S))\binom{m}{2}} \\ &= \exp\{\gamma m\sigma(S) - \varepsilon(m)\} \quad \text{where} \quad 0 \le \varepsilon(m) \le \frac{0.1}{m-1} \end{aligned}$$

(in the upper bound for $\varepsilon(m)$ we use that $\gamma < 0.3$). Thus we are in the situation described in Section 1.1.

1.5 Comparison with results in the literature

The topic of this paper touches upon some very active research areas, such as finding dense subgraphs in a given graph (see, for example, [7] and references therein), computational complexity of partition functions (see, for example, [10] and references therein) and complex zeros of partition functions (see, for example, [20] and references therein). Detecting dense subsets is notoriously hard. It is an NP-hard problem to approximate the size $|S|$ of the largest clique S within a factor of $|V|^{1-\varepsilon}$ for any fixed $0 < \varepsilon \le 1$ [14], [23]. In [11], a polynomial time algorithm for approximating the highest density of an m-subset within a factor of $O(n^{1/3-\varepsilon})$ for some small $\varepsilon > 0$ was constructed. The paper [8] presents an algorithm of $n^{O(1/\varepsilon)}$ complexity which approximates the highest density of an m-subset within a factor of $n^{1/4+\varepsilon}$ and an algorithm of $O\left(n^{\ln n}\right)$ complexity which approximates the density within a factor of $O(n^{1/4})$, the current record (note that [11], [7] and [8] normalize density slightly differently than we do). It has also been established that modulo some plausible complexity assumptions, it is hard to approximate the highest density of an m-subset within a constant factor, fixed in advance, see [7].

We note that while searching for a dense m-subset in a graph on n vertices, the most interesting case to consider is that of a growing m such that $m = o(n)$, since for any fixed $\varepsilon > 0$ one can compute in polynomial time the maximum number of edges of G spanned by an m-subset S of vertices of G within an additive error of εn^2 [12] (the complexity of the algorithm is exponential in $1/\varepsilon$). Of course, for any constant m, fixed in advance, all subsets of size m can be found in polynomial time by an exhaustive search.

The appearance of partition functions in combinatorics can be traced back to the work of Kasteleyn [17] on the statistics of dimers (perfect matchings), see also Section 8.7 of [19]. A randomized polynomial time approximation algorithm (based on the Markov Chain Monte Carlo approach) for computing the partition function of the classical Ising model was constructed by Jerrum and Sinclair in [16]. A lot of work has recently been done on understanding the computational complexity of the partition function of graph homomorphisms [10], where a certain "dichotomy" of instances has been established: it is either #P-hard to compute exactly (in most interesting cases) or it is computable in polynomial time exactly (in few cases). Recently, an approach inspired by the "correlation decay" phenomenon in statistical physics was used to construct deterministic approximation algorithms for partition functions in combinatorial problems [21], [2], [6]. To the best of author's knowledge, the partition function $P_m(W)$ introduced in Section 1.2 of this paper has not been studied before and the idea of our algorithm sketched in Section 1.3 is also new. On the other hand, one can view our method as inspired by the intuition from statistical physics. Essentially, the algorithm computes the partition function similar to the one used in the “simulated annealing" method, see [1], for the temperature that is provably higher than the phase transition threshold (the role of the “temperature" is played by $1/\gamma m$, where γ is the constant in the density functional (1.1)). The algorithm uses the fact that the partition function has nice analytic properties at temperatures above the phase transition temperature.

The corollary asserting that graphs without dense m-subsets can be efficiently separated from graphs with many dense m-subsets is vaguely similar in spirit to the property testing approach [13].

The result requiring most work is Theorem 1.2 bounding the complex roots of the partition function away from the matrix of all 1s. Studying complex roots of partition functions was initiated by Lee and Yang [22], [18], continued by Heilmann and Lieb [15], and it is a classical topic by now, because of its importance to locating the phase transition, see, for example, [20] and Section 8.5 of [19].

The general approach of this paper was first applied by the author [3] to approximate permanents of matrices that are sufficiently close to the matrix of all 1s (and related expressions, such as hafnians of symmetric matrices and multidimensional analogues of permanents of tensors). After the first version of this paper was written, P. Soberón and the author applied this approach to computing the partition function of graph homomorphisms [5] and its refinement [4]. In each case, the main work consists in obtaining a version of Theorem 1.2, which bounds the complex roots of the polynomial in question. It is the easiest in the case of permanents [3] and the most cumbersome in the case of graph homomorphisms with multiplicities [4]. The most general framework under which the method of this paper works is still at large.

Open Question 1.3. Is it true that for *any* $\gamma > 0$, fixed in advance, the density functional (1.1) can be computed (within a relative error of 0.1, say, and some $\varepsilon(m) = O(1/m)$) in $n^{O(\ln m)}$ time?

2 The algorithm

2.1 The algorithm for approximating the partition function

Given an $n \times n$ symmetric matrix $W = (w_{ij})$, we present an algorithm that computes the right hand side of the approximation (1.5) for the function $f(t)$ defined by formula (1.4).

Let

$$g(t) = P_m\big(J + t(W - J)\big) = \sum_{\substack{S \subset \{1,\dots,n\} \\ |S| = m}} \prod_{\substack{\{i,j\} \subset S \\ i \neq j}} \big(1 + t(w_{ij} - 1)\big), \tag{2.1}$$

so $f(t) = \ln g(t)$. Hence

$$f'(t) = \frac{g'(t)}{g(t)} \quad \text{and} \quad g'(t) = g(t) f'(t).$$

Therefore, for $k \geq 1$ we have

$$\frac{d^k}{dt^k} g(t)\Big|_{t=0} = \sum_{j=0}^{k-1} \binom{k-1}{j} \left(\frac{d^j}{dt^j} g(t)\Big|_{t=0}\right) \left(\frac{d^{k-j}}{dt^{k-j}} f(t)\Big|_{t=0}\right) \tag{2.2}$$

(we agree that the 0-th derivative of g is g).

We note that $g(0) = \binom{n}{m}$. If we compute the values of

$$\frac{d^k}{dt^k} g(t)\Big|_{t=0} \quad \text{for} \quad k = 1, \ldots, \ell, \tag{2.3}$$

then the formulas (2.2) for $k = 1, \ldots, \ell$ provide a non-degenerate triangular system of linear equations that allows us to compute

$$\frac{d^k}{dt^k} f(t)\Big|_{t=0} \quad \text{for} \quad k = 1, \ldots, \ell.$$

Hence our goal is to compute the values (2.3).

We have

$$\frac{d^k}{dt^k} g(t)\Big|_{t=0} = \sum_{\substack{S \subset \{1,\ldots,n\} \\ |S|=m}} \sum_{\substack{I=(\{i_1,j_1\},\ldots,\{i_k,j_k\}) \\ \{i_1,j_1\},\ldots,\{i_k,j_k\} \subset S}} (w_{i_1 j_1} - 1) \cdots (w_{i_k j_k} - 1)$$

where the inner sum is taken over all ordered sets I of k distinct pairs $\{i_1, j_1\}$, …, $\{i_k, j_k\}$ of points from S.

For an ordered set

$$I = (\{i_1, j_1\}, \ldots, \{i_k, j_k\})$$

of k pairs of points from the set $\{1, \ldots, n\}$, let

$$\rho(I) = \left| \bigcup_{s=1}^{k} \{i_s, j_s\} \right|$$

be the total number of distinct points among $i_1, j_1, \ldots, i_k, j_k$. Since there are exactly

$$\binom{n - \rho(I)}{m - \rho(I)}$$

m-subsets $S \subset \{1, \ldots, n\}$ containing the pairs $\{i_1, j_1\}$, …, $\{i_k, j_k\}$, we can further rewrite

$$\frac{d^k}{dt^k} g(t)\Big|_{t=0} = \sum_{\substack{I=(\{i_1,j_1\},\ldots,\{i_k,j_k\}) \\ \{i_1,j_1\},\ldots,\{i_k,j_k\} \subset \{1,\ldots,n\}}} \binom{n - \rho(I)}{m - \rho(I)} (w_{i_1 j_1} - 1) \cdots (w_{i_k j_k} - 1),$$

where the sum is taken over all ordered sets I of k distinct pairs $\{i_1, j_1\}, \ldots, \{i_k, j_k\}$ of points from the set $\{1, \ldots, n\}$. The algorithm consists in computing the latter sum. Since the sum contains not more than $n^{2k} = n^{O(\ell)}$ terms, the complexity of the algorithm is indeed $n^{O(\ell)}$, as claimed.

2.2 Proof of Lemma 1.1

The function $g(z)$ defined by the equation (2.1) is a polynomial in z of degree $d \leq \binom{m}{2}$ with $g(0) = \binom{n}{m} \neq 0$, so we factor

$$g(z) = g(0) \prod_{i=1}^{d} \left(1 - \frac{z}{\alpha_i}\right),$$

where $\alpha_1,\dots,\alpha_d$ are the roots of $g(z)$. By the condition of Lemma 1.1, we have

$$|\alpha_i| \;\geq\; \beta > 1 \quad \text{for} \quad i=1,\dots,d\,.$$

Therefore,

$$f(z)=\ln g(z)=\ln g(0)+\sum_{i=1}^{d}\ln\left(1-\frac{z}{\alpha_i}\right) \quad \text{provided} \quad |z|\leq 1\,, \tag{2.4}$$

where we choose the branch of $\ln g(z)$ that is real at $z=0$. Using the standard Taylor expansion, we obtain

$$\ln\left(1-\frac{1}{\alpha_i}\right)=-\sum_{k=1}^{\ell}\frac{1}{k}\left(\frac{1}{\alpha_i}\right)^k+\zeta_\ell\,,$$

where

$$|\zeta_\ell|=\left|\sum_{k=\ell+1}^{+\infty}\frac{1}{k}\left(\frac{1}{\alpha_i}\right)^k\right| \;\leq\; \frac{1}{(\ell+1)\beta^\ell(\beta-1)}\,.$$

Therefore, from the equation (2.4) we obtain

$$f(1)=f(0)+\sum_{k=1}^{\ell}\left(-\frac{1}{k}\sum_{i=1}^{d}\left(\frac{1}{\alpha_i}\right)^k\right)+\eta_\ell\,,$$

where

$$|\eta_\ell| \;\leq\; \frac{m(m-1)}{2(\ell+1)\beta^\ell(\beta-1)}\,.$$

It remains to notice that

$$-\frac{1}{k}\sum_{i=1}^{d}\left(\frac{1}{\alpha_i}\right)^k=\frac{1}{k!}\frac{d^k}{dt^k}f(t)\Big|_{t=0}\,.$$

□

3 Proof of Theorem 1.2

3.1 Definitions

For $\delta>0$, we denote by $\mathcal{U}(\delta)\subset\mathbb{C}^{n(n-1)/2}$ the closed polydisc of weights

$$\mathcal{U}(\delta)=\Big\{Z=(z_{ij})\,:\,\big|z_{ij}-1\big|\leq\delta \quad \text{for all} \quad 1\leq i\neq j\leq n\Big\}\,.$$

For a subset $\Omega\subset\{1,\dots,n\}$ where $0\leq|\Omega|\leq m$, we define

$$P_\Omega(Z)=\sum_{\substack{S\subset\{1,\dots,n\}\\|S|=m\\S\supset\Omega}}\;\prod_{\substack{\{i,j\}\subset S\\i\neq j}} z_{ij}\,.$$

In words: $P_\Omega(Z)$ is the restriction of the sum defining the clique partition function to the subsets S that contain a given set Ω. In particular,

$$P_\emptyset(Z)=P_m(Z)\,,$$

and if $|\Omega| < m$ then

$$P_\Omega(Z) = \frac{1}{m - |\Omega|} \sum_{i:\ i \notin \Omega} P_{\Omega \cup \{i\}}(Z)\,. \tag{3.1}$$

We note that the natural action of the symmetric group S_n on $\{1, \ldots, n\}$ induces the natural action of S_n on the space of polynomials in Z which permutes the polynomials $P_\Omega(Z)$.

First, we establish a simple geometric lemma.

Lemma 3.1. *Let $u_1, \ldots, u_n \in \mathbb{R}^2$ be non-zero vectors such that for some $0 \le \alpha < 2\pi/3$ the angle between any two vectors u_i and u_j does not exceed α. Let $u = u_1 + \cdots + u_n$. Then*

$$\|u\| \ \ge\ \left(\cos\frac{\alpha}{2}\right) \sum_{i=1}^{n} \|u_i\|\,.$$

Proof. We observe that the origin does not lie in the convex hull of any three vectors u_i, u_j, u_k since otherwise the angle between some two of these three vectors is at least $2\pi/3$. The Carathéodory Theorem then implies that the convex hull of the whole set of vectors $u_1, \ldots, u_n$ does not contain the origin and hence the vectors lie in an angle at most α with vertex at the origin. The length of the orthogonal projection of each vector u_i onto the bisector of the angle is at least $\|u_i\| \cos(\alpha/2)$ and hence the length of the orthogonal projection of $u = u_1 + \cdots + u_n$ onto the bisector of the angle is at least $(\|u_1\| + \cdots + \|u_n\|)\cos(\alpha/2)$. Since the length of the vector u is at least as large as the length of its orthogonal projection, the proof follows. □

Lemma 3.1 and its proof was suggested by B. Bukh [9]. It replaces an earlier weaker bound used by the author, see Lemma 3.1 of [3].

We prove Theorem 1.2 by reverse induction on $|\Omega|$, using Lemma 3.1 and the following two lemmas.

Lemma 3.2. *Let us fix real $0 < \tau < 1$, real $\delta > 0$ and an integer $1 \le r \le m$. Suppose that for all $\Omega \subset \{1, \ldots, n\}$ such that $|\Omega| = r$, for all $Z \in \mathcal{U}(\delta)$, we have $P_\Omega(Z) \ne 0$ and that for all $i = 1, \ldots, n$ we have*

$$|P_\Omega(Z)| \ \ge\ \frac{\tau}{m-1} \sum_{j:\ j \ne i} |z_{ij}| \left| \frac{\partial}{\partial z_{ij}} P_\Omega(Z) \right|.$$

Then, for any pair of subsets

$$\Omega_1, \Omega_2 \subset \{1, \ldots, n\} \quad \textit{such that} \quad |\Omega_1| = |\Omega_2| = r \quad \textit{and} \quad |\Omega_1 \triangle \Omega_2| = 2,$$

and for any $Z \in \mathcal{U}(\delta)$, the angle between the complex numbers $P_{\Omega_1}(Z)$ and $P_{\Omega_2}(Z)$, interpreted as vectors in $\mathbb{R}^2 = \mathbb{C}$, does not exceed

$$\theta = \frac{4\delta(m-1)}{\tau(1-\delta)}$$

and the ratio of $|P_{\Omega_1}(Z)|$ and $|P_{\Omega_2}(Z)|$ does not exceed

$$\lambda = \exp\left\{\frac{4\delta(m-1)}{\tau(1-\delta)}\right\}.$$

Proof. Let us choose an $\Omega \in \{1,\dots,n\}$ such that $|\Omega| = r$. Since $P_\Omega(Z) \neq 0$ for all $Z \in \mathcal{U}(\delta)$, we can choose a branch of $\ln P_\Omega(Z)$ that is a real number when $Z = J$. Then

$$\frac{\partial}{\partial z_{ij}} \ln P_\Omega(Z) = \left(\frac{\partial}{\partial z_{ij}} P_\Omega(Z)\right) \Big/ P_\Omega(Z).$$

Since $\left|z_{ij}\right| \geq 1-\delta$ for all $Z \in \mathcal{U}(\delta)$, we have

$$\sum_{j:\ j\neq i} \left|\frac{\partial}{\partial z_{ij}} \ln P_\Omega(Z)\right| \leq \frac{m-1}{\tau(1-\delta)} \quad \text{for} \quad i = 1,\dots,n. \tag{3.2}$$

Without loss of generality, we assume that $1 \in \Omega_1$, $2 \notin \Omega_1$ and $\Omega_2 = \Omega_1 \setminus \{1\} \cup \{2\}$.

Given $A \in \mathcal{U}(\delta)$, let $B \in \mathcal{U}(\delta)$ be the weights defined by

$$b_{1j} = a_{2j} \quad \text{and} \quad b_{2j} = a_{j1} \quad \text{for all} \quad j \neq 1,2$$

and

$$b_{ij} = a_{ij} \quad \text{for all other} \quad i,j.$$

Then

$$P_{\Omega_2}(A) = P_{\Omega_1}(B)$$

and

$$\begin{aligned}
|\ln P_{\Omega_1}(A) - \ln P_{\Omega_2}(A)| &= |\ln P_{\Omega_1}(A) - \ln P_{\Omega_1}(B)| \\
&\leq \left(\sup_{Z\in\mathcal{U}(\delta)} \sum_{j:\ j\neq 1,2} \left|\frac{\partial}{\partial z_{1j}} \ln P_{\Omega_1}(Z)\right| + \sum_{j:\ j\neq 1,2} \left|\frac{\partial}{\partial z_{2j}} \ln P_{\Omega_1}(Z)\right|\right) \\
&\quad \times \max_{j:\ j\neq 1,2} \left\{\left|a_{1j} - b_{1j}\right|, \left|a_{2j} - b_{2j}\right|\right\}.
\end{aligned}$$

Since $\left|a_{ij} - b_{ij}\right| \leq 2\delta$ for all $A,B \in \mathcal{U}(\delta)$, by the inequality (3.2) we conclude that the angle between the complex numbers $P_{\Omega_1}(A)$ and $P_{\Omega_2}(A)$ does not exceed θ and that the ratio of their absolute values does not exceed λ. □

Lemma 3.3. *Let us fix real $\delta > 0$, real $0 \leq \theta < 2\pi/3$, real $\lambda > 1$ and an integer $1 < r \leq m$. Suppose that for all $\Omega \subset \{1,\dots,n\}$ such that $m \geq |\Omega| \geq r$ and for all $Z \in \mathcal{U}(\delta)$, we have $P_\Omega(Z) \neq 0$ and that for any pair of subsets*

$$\Omega_1, \Omega_2 \subset \{1,\dots,n\} \quad \textit{such that} \quad m \geq |\Omega_1| = |\Omega_2| \geq r \quad \textit{and} \quad |\Omega_1 \triangle \Omega_2| = 2,$$

and for any $Z \in \mathcal{U}(\delta)$, the angle between two complex numbers $P_{\Omega_1}(Z)$ and $P_{\Omega_2}(Z)$ considered as vectors in $\mathbb{R}^2 = \mathbb{C}$ does not exceed θ while the ratio of $|P_{\Omega_1}(Z)|$ and $|P_{\Omega_2}(Z)|$ does not exceed λ. Let

$$\tau = \frac{\cos^2(\theta/2)}{\lambda}.$$

Then, for any subset $\Omega \subset \{1,\ldots,n\}$ *such that* $|\Omega| = r-1$*, all* $Z \in \mathcal{U}(\delta)$ *and all* $i = 1,\ldots,n$*, we have*

$$|P_\Omega(Z)| \;\geq\; \frac{\tau}{m-1} \sum_{j:\ j\neq i} |z_{ij}| \left| \frac{\partial}{\partial z_{ij}} P_\Omega(Z) \right|. \tag{3.3}$$

In addition,

$$\text{if} \quad \frac{n}{m} \;\geq\; \frac{\lambda}{\cos(\theta/2)},$$

the inequality (3.3) holds with

$$\tau = \cos(\theta/2).$$

Proof. Let us choose a subset $\Omega \subset \{1,\ldots,n\}$ such that $|\Omega| = r-1$. Let us define

$$\Omega_j = \Omega \cup \{j\} \quad \text{for} \quad j \notin \Omega.$$

Suppose first that $i \in \Omega$. Then, for $j \in \Omega \setminus \{i\}$ we have

$$\frac{\partial}{\partial z_{ij}} P_\Omega(Z) = z_{ij}^{-1} P_\Omega(Z)$$

and hence

$$|z_{ij}| \left| \frac{\partial}{\partial z_{ij}} P_\Omega(Z) \right| = |P_\Omega(Z)| \quad \text{provided} \quad i,j \in \Omega, \quad i \neq j.$$

For $j \notin \Omega$, we have

$$\frac{\partial}{\partial z_{ij}} P_\Omega(Z) = \frac{\partial}{\partial z_{ij}} P_{\Omega_j}(Z) = z_{ij}^{-1} P_{\Omega_j}(Z)$$

and hence

$$|z_{ij}| \left| \frac{\partial}{\partial z_{ij}} P_\Omega(Z) \right| = \left|P_{\Omega_j}(Z)\right| \quad \text{provided} \quad i \in \Omega \quad \text{and} \quad j \notin \Omega.$$

Summarizing,

$$\sum_{j:\ j\neq i} |z_{ij}| \left| \frac{\partial}{\partial z_{ij}} P_\Omega(Z) \right| = (r-2)\,|P_\Omega(Z)| + \sum_{j\notin\Omega} \left|P_{\Omega_j}(Z)\right|. \tag{3.4}$$

On the other hand, by the equation (3.1), we have

$$P_\Omega(Z) = \frac{1}{m-r+1} \sum_{j:\ j\notin\Omega} P_{\Omega_j}(Z).$$

By Lemma 3.1, we have

$$|P_\Omega(Z)| \;\geq\; \frac{\cos(\theta/2)}{m-r+1} \sum_{j:\ j\notin\Omega} \left|P_{\Omega_j}(Z)\right| \tag{3.5}$$

and

$$(m-1)\,|P_\Omega(Z)| \;\geq\; (r-2)\,|P_\Omega(Z)| + \left(\cos\frac{\theta}{2}\right) \sum_{j:\ j\notin\Omega} \left|P_{\Omega_j}(Z)\right|$$

and the inequality (3.3) with $\tau=\cos(\theta/2)$ follows by the formula (3.4).

Suppose now that $i \notin \Omega$. Then, for $j \in \Omega$, we have

$$\frac{\partial}{\partial z_{ij}} P_{\Omega}(Z) = \frac{\partial}{\partial z_{ij}} P_{\Omega_i}(Z) = z_{ij}^{-1} P_{\Omega_i}(Z)$$

and hence

$$|z_{ij}| \left| \frac{\partial}{\partial z_{ij}} P_{\Omega}(Z) \right| = |P_{\Omega_i}(Z)| \quad \text{provided} \quad i \notin \Omega \quad \text{and} \quad j \in \Omega\,. \tag{3.6}$$

If $r=m$ then

$$\frac{\partial}{\partial z_{ij}} P_{\Omega}(Z) = 0 \quad \text{for any} \quad j \notin \Omega$$

and

$$\sum_{j:\ j\neq i} |z_{ij}| \left| \frac{\partial}{\partial z_{ij}} P_{\Omega}(Z) \right| = (m-1)\,|P_{\Omega_i}|\,.$$

By the equation (3.1), we have

$$P_{\Omega}(Z) = \sum_{j \notin \Omega} P_{\Omega_j}(Z)$$

and hence by Lemma 3.1

$$|P_{\Omega}(Z)| \geq \left(\cos\frac{\theta}{2}\right) \sum_{j\notin\Omega} \left|P_{\Omega_j}(Z)\right| \geq \left(\cos\frac{\theta}{2}\right) |P_{\Omega_i}(Z)|\,,$$

which proves the inequality (3.3) in the case of $r=m$ with $\tau=\cos(\theta/2)$.

If $r<m$, then for $j\notin\Omega_i$, let $\Omega_{ij}=\Omega\cup\{i,j\}$. Then, for $j\notin\Omega_i$, we have

$$\frac{\partial}{\partial z_{ij}} P_{\Omega}(Z) = \frac{\partial}{\partial z_{ij}} P_{\Omega_{ij}}(Z) = z_{ij}^{-1} P_{\Omega_{ij}}(Z)$$

and hence

$$|z_{ij}| \left| \frac{\partial}{\partial z_{ij}} P_{\Omega}(Z) \right| = \left|P_{\Omega_{ij}}(Z)\right| \quad \text{provided} \quad i \notin \Omega \quad \text{and} \quad j \notin \Omega_i\,.$$

From the formula (3.6),

$$\sum_{j:\ j\neq i} |z_{ij}| \left| \frac{\partial}{\partial z_{ij}} P_{\Omega}(Z) \right| = (r-1)\,|P_{\Omega_i}(Z)| + \sum_{j\notin\Omega_i} \left|P_{\Omega_{ij}}(Z)\right| \quad \text{if} \quad r<m\,. \tag{3.7}$$

By the equation (3.1), we have

$$P_{\Omega_i}(Z) = \frac{1}{m-r} \sum_{j:\ j\notin\Omega_i} P_{\Omega_{ij}}(Z)$$

and hence by Lemma 3.1, we have

$$|P_{\Omega_i}(Z)| \geq \frac{\cos(\theta/2)}{m-r} \sum_{j:\ j\notin\Omega_i} \left|P_{\Omega_{ij}}(Z)\right|.$$

Comparing this with the equation (3.7), we conclude as above that

$$|P_{\Omega_i}(Z)| \;\geq\; \frac{\cos(\theta/2)}{m-1} \sum_{j:\; j\neq i} |z_{ij}| \left| \frac{\partial}{\partial z_{ij}} P_{\Omega}(Z) \right| . \tag{3.8}$$

It remains to compare $|P_{\Omega_i}(Z)|$ and $|P_{\Omega}(Z)|$. Since the ratio of any two $\left|P_{\Omega_{j_1}}(Z)\right|$, $\left|P_{\Omega_{j_2}}(Z)\right|$ does not exceed λ, from the inequality (3.5) we conclude that

$$|P_{\Omega}(Z)| \;\geq\; \frac{\cos(\theta/2)}{\lambda} \frac{n-r+1}{m-r+1} |P_{\Omega_i}(Z)| \tag{3.9}$$

$$\geq \frac{\cos(\theta/2)}{\lambda} |P_{\Omega_i}(Z)| \quad \text{for all} \quad i \notin \Omega . \tag{3.10}$$

The proof of the inequality (3.3) with $\tau = \dfrac{\cos^2(\theta/2)}{\lambda}$ follows by the inequality (3.8) and inequality (3.10). In addition, if

$$\frac{n}{m} \;\geq\; \frac{\lambda}{\cos(\theta/2)}$$

then

$$\frac{n-r+1}{m-r+1} \;\geq\; \frac{\lambda}{\cos(\theta/2)}$$

and the proof of the inequality (3.3) with $\tau = \cos(\theta/2)$ follows by the inequality (3.8) and the inequality (3.9). $\square$

3.2 Proof of Theorem 1.2

One can see that for a sufficiently small $\omega > 0$, the equation

$$\theta = \frac{4\omega \exp\{\theta\}}{(1-\omega)\cos^2(\theta/2)}$$

has a solution $0 < \theta < 2\pi/3$. Numerical computations show that one can choose $\omega = 0.071$, in which case

$$\theta \approx 0.6681776075 .$$

Let

$$\lambda = \exp\{\theta\} \approx 1.950679176 \quad \text{and} \quad \tau = \frac{\cos^2(\theta/2)}{\exp\{\theta\}} \approx 0.4575206588 .$$

Let

$$\delta = \frac{\omega}{m-1} .$$

For $r = m, \ldots, 1$ we prove the following Statement 3.4, Statement 3.5 and Statement 3.6:

Statement 3.4. *Let $\Omega \subset \{1, \ldots, n\}$ be a set such that $|\Omega| = r$. Then for any $Z \in \mathcal{U}(\delta)$ we have*

$$P_{\Omega}(Z) \neq 0 .$$

Statement 3.5. *Let $\Omega \subset \{1,\ldots,n\}$ be a set such that $|\Omega| = r$. Then for any $Z \in \mathcal{U}(\delta)$, $Z = (z_{ij})$, and any $i = 1,\ldots,n$, we have*

$$|P_\Omega(Z)| \ \geq \ \frac{\tau}{m-1} \sum_{j:\ j\neq i} |z_{ij}| \left| \frac{\partial}{\partial z_{ij}} P_\Omega(Z) \right|.$$

Statement 3.6. *Let $\Omega_1, \Omega_2 \subset \{1,\ldots,n\}$ be sets such that $|\Omega_1| = |\Omega_2| = r$ and $|\Omega_1 \triangle \Omega_2| = 2$. Then the angle between the complex numbers $P_{\Omega_1}(Z)$ and $P_{\Omega_2}(Z)$, considered as vectors in $\mathbb{R}^2 = \mathbb{C}$, does not exceed θ, whereas the ratio of $|P_{\Omega_1}(Z)|$ and $|P_{\Omega_2}(Z)|$ does not exceed λ.*

Suppose that $r = m$ and let $\Omega \subset \{1,\ldots,n\}$ be a subset such that $|\Omega| = m$. Then

$$P_\Omega(Z) = \prod_{\substack{\{i,j\}\subset\Omega \\ i\neq j}} z_{ij}\,,$$

so Statement 3.4 clearly holds for $r = m$. Moreover, for $i \in \Omega$, we have

$$\sum_{j:\ j\neq i} |z_{ij}| \left| \frac{\partial}{\partial z_{ij}} P_\Omega(Z) \right| = (m-1)\,|P_\Omega(Z)|\,,$$

while for $i \notin \Omega$, we have

$$\sum_{j:\ j\neq i} |z_{ij}| \left| \frac{\partial}{\partial z_{ij}} P_\Omega(Z) \right| = 0\,,$$

so Statement 3.5 holds for $r = m$ as well.

Lemma 3.2 implies that if Statement 3.4 and Statement 3.5 hold for sets Ω of cardinality r then for any two subsets $\Omega_1, \Omega_2 \subset \{1,\ldots,n\}$ such that $|\Omega_1| = |\Omega_2| = r$ and $|\Omega_1 \triangle \Omega_2| = 2$ the angle between the two (necessarily non-zero) complex numbers $P_{\Omega_1}(Z)$ and $P_{\Omega_2}(Z)$ does not exceed

$$\frac{4\delta(m-1)}{\tau(1-\delta)} = \frac{4\omega}{\left(1-\frac{\omega}{m-1}\right)\tau} = \frac{4\omega\exp\{\theta\}}{\left(1-\frac{\omega}{m-1}\right)\cos^2(\theta/2)} \ \leq \ \frac{4\omega\exp\{\theta\}}{(1-\omega)\cos^2(\theta/2)} = \theta$$

while the ratio of $|P_{\Omega_1}(Z)|$ and $|P_{\Omega_2}(Z)|$ does not exceed

$$\begin{aligned}\exp\left\{\frac{4\delta(m-1)}{\tau(1-\delta)}\right\} \ &= \ \exp\left\{\frac{4\omega\exp\{\theta\}}{\left(1-\frac{\omega}{m-1}\right)\cos^2(\theta/2)}\right\} \ \leq \ \exp\left\{\frac{4\omega\exp\{\theta\}}{(1-\omega)\cos^2(\theta/2)}\right\} \\ &= \ \exp\{\theta\} \ = \ \lambda\end{aligned}$$

and hence Statement 3.6 holds for sets Ω of cardinality r.

Lemma 3.3 implies that if Statement 3.4 and Statement 3.6 hold for sets Ω of cardinality r then Statement 3.5 holds for sets Ω of cardinality $r-1$. Formula (3.1), Lemma 3.1, Statement 3.4 and Statement 3.6 for sets Ω of cardinality r imply Statement 3.4 for sets Ω of cardinality $r-1$. Therefore, we conclude that Statement 3.4 and Statement 3.6 hold for sets Ω such that $|\Omega| = 1$. Formula (3.1) and Lemma 3.1 imply then that

$$P_\emptyset(Z) = P_m(Z) \neq 0\,,$$

as desired.

A more careful analysis establishes that if $m \geq 10$ then for $\omega = 0.271$ the equation

$$\theta = \frac{4\omega}{\left(1 - \frac{\omega}{m-1}\right)\cos(\theta/2)}$$

has a solution

$$0 < \theta < 1.626699443$$

with

$$\lambda = \exp\{\theta\} < 5.1 \quad \text{and} \quad \cos(\theta/2) > 0.68 \quad \text{so that} \quad \frac{\lambda}{\cos(\theta/2)} < 7.5\,.$$

Then, if $n \geq 8m$, we let

$$\delta = \frac{\omega}{m-1}, \quad \tau = \cos(\theta/2)$$

and the proof proceeds as above. □

Remark 3.7. As follows from Statement 3.4, we have $P_\Omega(Z) \neq 0$ for any complex weights $Z = (z_{ij})$ satisfying the conditions of Theorem 1.2. The algorithm of Section 2 extends in a straightforward way to computing $P_\Omega(W)$ for every weights $W = (w_{ij})$ satisfying the inequality (1.3). This allows us to compute in $n^{O(\ln m)}$ time by successive conditioning an m-subset S which satisfies the inequality (1.2). Namely, we define weights W by the formula (1.6) and construct subsets $S_0, \ldots, S_m \subset \{1, \ldots, n\}$, where $S_0 = \emptyset$ and $|S_i| = i$ as follows. Assuming that S_{i-1} is constructed, for each $j \in \{1, \ldots, n\} \setminus S_{i-1}$, we let $\Omega_j = S_{i-1} \cup \{j\}$ and compute $P_{\Omega_j}(W)$ within a relative error of $1/10m$. We then select j with the largest computed value of $P_{\Omega_j}(W)$ and let $S_i = \Omega_j$. The set $S = S_m$ satisfies the inequality (1.2).

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