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NOTE

Groups with Identical k-Profiles

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Abstract: We show that for $1 \leq k \leq \sqrt{2\log_3 n} - (5/2)$, the multiset of isomorphism types of k-generated subgroups does not determine a group of order at most n. This answers a question raised by Tim Gowers in connection with the Group Isomorphism problem.

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1 Introduction

We say that a group is k-generated if it has a set of at most k generators. Let $\mathcal{G}_k$ be the set of isomorphism types[1] of all k-generated finite groups. Let G be a finite group. Following Gowers [3], we say that the *k-profile* of G is the function $f_G : \mathcal{G}_k \to \mathbb{N}$ defined by letting $f_G(H)$ be the number of subgroups of G isomorphic to H ($H \in \mathcal{G}_k$).

Tim Gowers raised the question [3], for which k does the k-profile determine a group of order n? Such a k yields a simple isomorphism test[2] in time $n^{O(k)}$ for groups of order n given by their Cayley tables (see Section 3).

[1] Two groups belong to the same *isomorphism type* if and only if they are isomorphic.

[2] Regarding the significance of the Group Isomorphism problem to the Graph Isomorphism problem we refer the reader to Section 13 of [1] and especially to footnote 9 in that section.

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Theorem 1.1. *If p is an odd prime, k and n are positive integers, and*

$$1 \le k \le \sqrt{2\log n/\log p} - (5/2)\,,$$

then there exist nonisomorphic p-groups of order at most n with identical k-profiles.

Remark 1.2. In particular, setting $p = 3$, we see that if k and n are positive integers such that $1 \le k \le \sqrt{2\log_3 n} - (5/2)$, then there exist nonisomorphic groups of order at most n with identical k-profiles.

Our examples are p-groups of class 2 and exponent p.

Theorem 1.3. *For any odd prime p and positive integer k there exist nonisomorphic p-groups of class 2, exponent p, and order p^N, where $N = (k+2)(k+3)/2$, with identical k-profiles.*

2 The proof

Recall that a nilpotent group G is of *class 2* if $G' \le Z(G)$, where G' denotes the commutator subgroup $G' = [G,G]$ and $Z(G)$ denotes the center of G. For an odd prime p, a *relatively free* p-group P of class 2 and exponent p with m generators can be obtained from a free group with m generators by factoring out all elements u^p and all commutators $[[u,v],w]$.

Fact 2.1. *For real numbers m and k such that $m \ge k+2$, we have*

$$m(m-1)/2 \ge 1 + mk - (k^2+k)/2\,.$$

Proof. Let $x = m-k-2$, so $x \ge 0$ and we wish to show that $f(x) \ge 0$ where

$$f(x) = (k+2+x)(k+1+x) - 2(k+2+x)k + k^2 + k - 2\,.$$

But then $f(x) = x^2 + 3x \ge 0$, as desired. □

Fact 2.2. *For an odd prime p and a positive integer k we have*

$$(p^k-1)(p^k-p)\cdots(p^k-p^{k-1}) > (1/2)p^{k^2}\,.$$

Proof.

$$\frac{\prod_{i=0}^{k-1}(p^k-p^i)}{p^{k^2}} = \prod_{j=1}^{k}\left(1-\frac{1}{p^j}\right) > 1 - \sum_{j=1}^{\infty}\frac{1}{p^j} = 1 - \frac{1}{p-1} \ge \frac{1}{2}\,.$$ □

Hypothesis 2.3.

(i) p is an odd prime,

(ii) m is a positive integer, and

(iii) P is a relatively free group with m generators, class two, and exponent p.

Lemma 2.4. *Assume Hypothesis 2.3. Suppose k is a positive integer such that $m \geq k+2$. Then there exists an element of P' that does not lie in Q' for any k-generated subgroup Q of P.*

Note. This is false for $k = 2$ and $m = k+1 = 3$.

Proof. In this situation, $P' = Z(P), |P/P'| = p^m$, and $|P'| = p^{m(m-1)/2}$.

We claim that for every k-generated subgroup Q of P, there exists a k-generated subgroup R of P such that $R' \geq Q'$ and $|R/(R \cap P')| = p^k$.

Indeed, let Q be a k-generated subgroup of P and $p^i = |Q/(Q \cap P')| = |QP'/P'|$. Let $s_1, \ldots, s_i$ be elements of Q such that $Q \cap P'$ together with $s_1, \ldots, s_i$ generate Q. Let $S = \langle s_1, \ldots, s_i \rangle$. Then $i \leq k$. If $i = k$, let $R = S$. If $i < k$ then there exist elements $s_{i+1}, \ldots, s_k$ such that $|RP'/P'| = p^k$ for $R = \langle s_1, \ldots, s_k \rangle$. In both cases, $|RP'/P'| = p^k$, $|R'| = p^{k(k-1)/2}$, $Q = S(Q \cap P') \leq SP' \leq RP'$, and $Q' \leq (RP')' = R'$. This proves the claim.

The number of distinct subgroups of the form RP' is the same as the number of k-dimensional subspaces of an m-dimensional vector space over the prime field $\mathbb{F}_p$. Call this number $N(m,k)$. Then

$$N(m,k) = \frac{(p^m - 1)(p^m - p) \ldots (p^m - p^{k-1})}{(p^k - 1)(p^k - p) \ldots (p^k - p^{k-1})}. \tag{1}$$

Clearly, the numerator of $N(m,k)$ is less than p^{mk}. By Fact 2.2, the denominator is greater than $(1/2)p^{k^2}$. Therefore, $N(m,k) < 2p^{mk-k^2}$. Since $p \geq 3$, we have $N(m,k) < p^{mk-k^2+1}$.

Now we count the elements of P' that lie in Q' for some k-generated subgroup Q of P. Each such element lies in $(RP')'$ for some subgroup RP' as above. So we obtain the upper bound

$$p^{k(k-1)/2} N(m,k) < p^{e+1} \tag{2}$$

for $e = (k^2 - k)/2 + mk - k^2 = mk - (k^2 + k)/2$.

We saw above that $|P'| = p^{m(m-1)/2}$. Fact 2.1 shows that

$$m(m-1)/2 \geq e + 1\,.$$

This gives the desired conclusion. □

Lemma 2.5. *Assume Hypothesis 2.3 for a group P_1 in place of P. Let d be a positive integer such that $m \geq d+2$. Let $P_2 = \langle w \rangle$ be a cyclic group of order p and $P = P_1 \times P_2$. Then there exists an element v of P_1' such that*

(a) $|\langle v, w \rangle| = p^2$,

(b) $P/\langle v \rangle$ *is not isomorphic to* $P/\langle w \rangle$*, and*

(c) for every d-generated subgroup Q of P we have $Q' \cap \langle v, w \rangle = 1$.

Proof. By Lemma 2.4, P_1' has an element v that does not lie in Q' for any d-generated subgroup Q of P. Then (a) is obvious. We obtain (b) because

$$(P/\langle v \rangle)' = P_1'/\langle v \rangle \quad \text{and} \quad (P/\langle w \rangle)' \cong P_1'\,. \tag{3}$$

To obtain (c), let $s_1,\dots,s_d$ be d elements of P. Set $R=\langle s_1,\dots,s_d\rangle$. Then there exist unique elements $u_1,\dots,u_d$ of P_1 such that $u_i^{-1}s_i\in\langle w\rangle$ for each i, and $R'=Q'$ where $Q=\langle u_1,\dots,u_d\rangle$. By the choice of v, we see that $v\notin R'$. As $R'\le P_1$, we have $R'\cap\langle v,w\rangle=1$. □

Lemma 2.6. *Assume the hypothesis and notation of Lemma 2.5. Then there exists a bijection between the set of all d-generated subgroups of $P/\langle v\rangle$ and the set of all d-generated subgroups of $P/\langle w\rangle$ such that corresponding subgroups are isomorphic.*

Proof. Consider a d-generated subgroup Q of $P/\langle v\rangle$. Then $Q=Q^*/\langle v\rangle$ for a subgroup Q^* of P that contains v, and $Q^*=\langle Q_0,v\rangle$ for some d-generated subgroup Q_0 of P. Let $Q^{**}=\langle Q^*,w\rangle=\langle Q_0,v,w\rangle$. Recall that v and w are in $Z(P)$. So

$$(Q^{**})'=(Q^*)'=(Q_0)'. \tag{4}$$

By Lemma 2.5 we infer $(Q^{**})'\cap\langle v,w\rangle=1$.

For a d-generated subgroup R of $P/\langle w\rangle$, we obtain analogous subgroups R^*, R_0, R^{**} of P. Note that Q and R uniquely determine Q^{**} and R^{**}.

Now consider the family of all subgroups S of P such that

(i) v and w are in S, and

(ii) $S=\langle S_0,v,w\rangle$ for some d-generated subgroup S_0 of S.

The analysis above shows that to prove Lemma 2.6, it suffices to obtain, for each subgroup S as above, a bijection between

- the set of all d-generated subgroups Q of $P/\langle v\rangle$ for which $Q^{**}=S$ and
- the set of all d-generated subgroups R of $P/\langle w\rangle$ for which $R^{**}=S$

such that corresponding subgroups Q and R are isomorphic.

For each subgroup S, we have $S'\cap\langle v,w\rangle=S_0'\cap\langle v,w\rangle=1$ by Lemma 2.5.

Since P has exponent p and S/S' is abelian, there exists a complement S_1/S' to $\langle S',v,w\rangle/S'$ in S/S'. Since S', v, and w are central, we have $S=S_1\times\langle v,w\rangle$. Therefore, there exists a unique automorphism of S that induces the identity on S_1 and switches v and w. This establishes the desired bijection. □

Proof of Theorem 1.3. The result is contained in Lemma 2.6. Let $m=k+2$. Then

$$|P|=p^{1+m(m+1)/2}=p^{1+(k+2)(k+3)/2}.$$

The groups $P/\langle v\rangle$ and $P/\langle w\rangle$ have order $|P|/p$. □

Proof of Theorem 1.1. The condition $k\le\sqrt{2\log n/\log p}-(5/2)$ means

$$n\ge p^{(k+(5/2))^2/2}>p^{(k+2)(k+3)/2}=p^N.$$

By Theorem 1.3, there exist nonisomorphic groups of order p^N with identical k-profiles. □

Remark 2.7. We comment on the case $k = 1$. It is obvious that p-groups of exponent p of equal order have the same 1-profile. In particular, for every odd prime p there exist nonisomorphic p-groups of order p^3 with the same 1-profile. Moreover, for all primes p there exists a nonabelian group of order p^4 with a cyclic subgroup of order p^3 called $M_4(p)$, which has the same 1-profile as the direct product of a cyclic group of order p^3 and the cyclic group of order p. (For the definition of $M_4(p)$ see the classification of p-groups with a cyclic subgroup of index p in [2, pp. 192–193].) In particular, $M_4(2)$ has order 16, improving Remark 1.2 for $k = 1$.

3 The isomorphism test

We describe the isomorphism test based on k-profiles suggested by Gowers [3].

Proposition 3.1. *Let k,n be positive integers and suppose the groups of order n are determined, up to isomorphism, by their k-profiles. Then isomorphism of two groups of order n, given by their Cayley tables, can be decided in time $n^{2k+O(1)}$.*

Proof. Let G,H be two groups of order n. By our assumption, G and H are isomorphic if and only if their k-profiles agree, so we only need to show how to compare the k-profiles of the two groups. This can be done by computing the following equivalence relation on the disjoint union $X := G^k \,\dot{\cup}\, H^k$. We say that two k-tuples $(x_1,\dots,x_k) \in X$ and $(y_1,\dots,y_k) \in X$ are equivalent if the correspondence $x_i \mapsto y_i$ extends to an isomorphism of the subgroups generated by these k-tuples. This can be checked in polynomial time per instance, so $n^{2k+O(1)}$ total time. Now the k-profiles of G and H agree if and only if each equivalence class is evenly divided between G^k and H^k. □

Remark 3.2. While our result shows that the comparison of k-profiles alone will not solve the Group Isomorphism problem in polynomial time, it does not rule out a role for this algorithm in improving the state of the art in this area. Indeed, Group Isomorphism is not currently known to be testable in time $n^{o(\log n)}$ (cf. [4, 6, 5, 7]). Therefore, if our bound on k is not very far from being tight, say the result stated in Remark 1.2 would fail if we replace $\sqrt{2\log_3 n}$ by $O((\log n)^{0.99})$, this would mean progress on the complexity of the Group Isomorphism problem.

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