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Approximation Algorithms for Hypergraph Small-Set Expansion and Small-Set Vertex Expansion

Anand Louis* Yury Makarychev†

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Abstract: The expansion of a hypergraph, a natural extension of the notion of expansion in graphs, is defined as the minimum over all cuts in the hypergraph of the ratio of the number of the hyperedges cut to the size of the smaller side of the cut. We study the Hypergraph Small-Set Expansion problem, which, for a parameter $\delta \in (0, 1/2]$, asks to compute the cut having the least expansion while having at most δ fraction of the vertices on the smaller side of the cut. We present two algorithms. Our first algorithm gives an $\widetilde{O}(\delta^{-1}\sqrt{\log n})$–approximation. The second algorithm finds a set with expansion $\widetilde{O}(\delta^{-1}(\sqrt{d_{\max} r^{-1} \log r\, \phi^*} + \phi^*))$ in an r–uniform hypergraph with maximum degree $d_{\max}$ (where ϕ^* is the expansion of the optimal solution). Using these results, we also obtain algorithms for the Small-Set Vertex Expansion

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problem: we get an $\widetilde{O}(\delta^{-1}\sqrt{\log n})$–approximation algorithm and an algorithm that finds a set with vertex expansion

$$\widetilde{O}\left(\delta^{-1}\sqrt{\phi^V \log d_{\max}} + \delta^{-1}\phi^V\right)$$

(where ϕ^V is the vertex expansion of the optimal solution).

For $\delta = 1/2$, Hypergraph Small-Set Expansion is equivalent to the hypergraph expansion problem. In this case, our approximation factor of $O(\sqrt{\log n})$ for expansion in hypergraphs matches the corresponding approximation factor for expansion in graphs due to Arora, Rao, and Vazirani (JACM 2009).

1 Introduction

The expansion of a hypergraph, a natural extension of the notion of expansion in graphs, is defined as follows.

Definition 1.1 (Hypergraph Expansion)**.** Given a hypergraph $H = (V,E)$ on n vertices (each edge $e \in E$ of H is a subset of vertices), we say that an edge $e \in E$ is cut by a set S if $e \cap S \neq \varnothing$ and $e \cap \bar{S} \neq \varnothing$ (i. e., some vertices in e lie in S and some vertices lie outside of S). We denote the set of edges cut by S by $E_{\text{cut}}(S)$. The expansion $\phi(S)$ of a set $S \subset V$ ($S \neq \varnothing$, $S \neq V$) in a hypergraph $H = (V,E)$ is defined as

$$\phi(S) = \frac{|E_{\text{cut}}(S)|}{\min(|S|,|\bar{S}|)}\,.$$

Hypergraph expansion and related hypergraph partitioning problems are of immense practical importance, having applications in parallel and distributed computing [6], VLSI circuit design and computer architecture [13], scientific computing [10] and other areas. In spite of this, there has not been much theoretical work on hypergraph partitioning problems. In this paper, we study a generalization of the Hypergraph Expansion problem, namely the Hypergraph Small-Set Expansion problem.

Problem 1.2 (Hypergraph Small-Set Expansion Problem)**.** Given a hypergraph $H = (V,E)$ and a parameter $\delta \in (0,1/2]$, the Hypergraph Small-Set Expansion problem (H-SSE) is to find a set $S \subset V$ of size at most δn that minimizes $\phi(S)$. The value of the optimal solution to H-SSE is called the *small-set expansion* of H. That is, for $\delta \in (0,1/2]$, the small-set expansion $\phi^*_{H,\delta}$ of a hypergraph $H = (V,E)$ is defined as

$$\phi^*_{H,\delta} = \min_{\substack{S\subset V\\ 0<|S|\leq\delta n}} \phi(S)\,.$$

Note that for $\delta = 1/2$, the Hypergraph Small-Set Expansion Problem is the Hypergraph Expansion Problem.

We note that H-SSE can be reduced to SSE (small-set expansion in graphs) if all hyperedges have bounded size. Let r be the size of the largest hyperedge in H. Construct an auxiliary (weighted) graph F on V as follows: pick a vertex in each hyperedge e and connect it in F to all other vertices of e (i. e., replace e with a star); let the weight of an edge f in F be the total weight of the hyperedges $e \in E$ for which f is part of the representative star of e. Then solve SSE in the graph F. It is easy to see that if we solve SSE using an α–approximation algorithm, then we get $(r-1)\alpha$–approximation for H-SSE. This approach gives $O(\sqrt{\log n \log(1/\delta)})$–approximation if r is bounded. However, if H is an arbitrary hypergraph, we only get an $O(n\sqrt{\log n \log(1/\delta)})$–approximation. The goal of this paper is to give an approximation guarantee valid for hypergraphs with hyperedges of arbitrary size.

Related work. Small-Set Expansion in graphs has attracted a lot of attention recently. The problem was introduced by Raghavendra and Steurer [22], who showed that it is closely related to the Unique Games problem. Raghavendra, Steurer and Tetali [23] designed an algorithm for SSE that finds a set of size $O(\delta n)$ with expansion $O(\sqrt{\phi^* d \log(1/\delta)})$ in d regular graphs (where ϕ^* is the expansion of the optimal solution). Later Bansal, Feige, Krauthgamer, Makarychev, Nagarajan, Naor, and Schwartz [5] gave a $O(\sqrt{\log n \log(1/\delta)})$–approximation algorithm for the problem. In a different direction, Louis [16] and Chan, Louis, Tang and Zhang [7] studied a heat (Markov) operator for hypergraphs and showed connections between its spectrum and some combinatorial properties of hypergraphs. In particular, they used some tools developed in this paper to show connections between the small-set expansion of a hypergraph and higher eigenvalues of its heat operator.

Our results. We present analogs of the results by Bansal et al. [5] and Raghavendra, Steurer and Tetali [23] for hypergraphs.

Theorem 1.3. *There is a randomized polynomial-time approximation algorithm for the Hypergraph Small-Set Expansion problem that, given a hypergraph $H=(V,E)$ and parameters $\varepsilon \in (0,1)$ and $\delta \in (0,1/2)$, finds a set $S \subset V$ of size at most $(1+\varepsilon)\delta n$ such that*

$$\phi(S) \le O_\varepsilon\left(\delta^{-1}\log\delta^{-1}\log\log\delta^{-1}\cdot\sqrt{\log n}\cdot\phi^*_{H,\delta}\right) = \widetilde{O}_\varepsilon\left(\delta^{-1}\sqrt{\log n}\,\phi^*_{H,\delta}\right),$$

(where the constant in the O notation depends polynomially on $1/\varepsilon$). That is, the algorithm gives $O(\sqrt{\log n})$–approximation when δ and ε are fixed.

We state our second result, Theorem 1.4, for r-uniform hypergraphs. We present and prove a more general Theorem 6.3 that applies to any hypergraphs in Section 6.

Theorem 1.4. *There is a randomized polynomial-time algorithm for the Hypergraph Small-Set Expansion problem that, given an r–uniform hypergraph $H=(V,E)$ with maximum degree $d_{\max}$ and parameters $\varepsilon \in (0,1)$ and $\delta \in (0,1/2)$, finds a set $S \subset V$ of size at most $(1+\varepsilon)\delta n$ such that*

$$\phi(S) \le \widetilde{O}_\varepsilon\left(\delta^{-1}\left(\sqrt{d_{\max}\frac{\log r}{r}\phi^*_{H,\delta}} + \phi^*_{H,\delta}\right)\right).$$

Our algorithms for H-SSE are bi-criteria approximation algorithms in that they output a set S of size at most $(1+\varepsilon)\delta n$. We note that this is similar to the algorithm by Bansal et al. [5] for SSE, which also finds a set of size at most $(1+\varepsilon)\delta n$ rather than a set of size at most δn. The algorithm by Raghavendra, Steurer and Tetali [23] finds a set of size $O(\delta n)$. The approximation factor of our first algorithm does not depend on the size of hyperedges in the input hypergraph. It has the same dependence on n as the algorithm by Bansal et al. [5] for SSE. However, the dependence on $1/\delta$ is quasi-linear; whereas it is logarithmic in the algorithm by Bansal et al. [5]. In fact, we show that the integrality gap of the standard SDP relaxation for H-SSE is at least linear in $1/\delta$ (Theorem 7.1). The approximation guarantee of our second algorithm is analogous to that of the algorithm by Raghavendra, Steurer and Tetali [23].

Small-Set Vertex Expansion. Our techniques can also be used to obtain an approximation algorithm for Small-Set Vertex Expansion (SSVE) in graphs.

Problem 1.5 (Small-Set Vertex Expansion Problem)**.** Given graph $G=(V,E)$, the vertex expansion of a set $S \subset V$ is defined as

$$\phi^V(S) = \frac{|\{u \in \bar{S} : \exists v \in S \text{ such that } \{u,v\} \in E\}|}{|S|}\,.$$

Given a parameter $\delta \in (0,1/2]$, the Small-Set Vertex Expansion problem (SSVE) is to find a set $S \subset V$ of size at most δn that minimizes $\phi^V(S)$. The value of the optimal solution to SSVE is called the small-set vertex expansion of G. That is, for $\delta \in (0,1/2]$, the small-set expansion $\phi^V_{G,\delta}$ of a graph $G=(V,E)$ is defined as

$$\phi^V_{G,\delta} = \min_{\substack{S\subset V\\ 0<|S|\leq \delta n}} \phi^V(S)\,.$$

Small-Set Vertex Expansion recently gained attention thanks to its connection to obtaining subexponential-time, constant-factor approximation algorithms for several combinatorial problems like Sparsest Cut and Graph Coloring [2, 19]. Using a reduction from vertex expansion in graphs to hypergraph expansion, we can get an approximation algorithm for SSVE having the same approximation guarantee as that for H-SSE.

Theorem 1.6 ("Graph to Hypergraph Theorem," informal statement)**.** *There exist absolute constants $c_1, c_2 \in \mathbb{R}^+$ such that for every graph $G=(V,E)$, there exists a polynomial-time computable hypergraph $H=(V',E')$ such that $c_1\phi^*_{H,\delta} \leq \phi^V_{G,\delta} \leq c_2\phi^*_{H,\delta}$. Furthermore, the algorithm for H-SSE from Theorem 6.3 applies to H.*

A detailed version of this statement appears as Theorem 8.2 in Section 8.

From this theorem, Theorem 1.3 and Theorem 6.3 we immediately get algorithms for SSVE.

Theorem 1.7 (Corollary to Theorem 1.3 and Theorem 8.2)**.** *There is a randomized polynomial-time approximation algorithm for the Small-Set Vertex Expansion problem that, given a graph $G=(V,E)$ and parameters $\varepsilon \in (0,1)$ and $\delta \in (0,1/2)$), finds a set $S \subset V$ of size at most $(1+\varepsilon)\delta n$ such that*

$$\phi^V(S) \leq O_\varepsilon\left(\sqrt{\log n}\,\delta^{-1}\log\delta^{-1}\log\log\delta^{-1}\cdot\phi^V_{G,\delta}\right)\,.$$

That is, the algorithm gives $O(\sqrt{\log n})$–approximation when δ and ε are fixed.

Theorem 1.8 (Corollary to Theorem 6.3 and Theorem 8.2)**.** *There is a randomized polynomial-time algorithm for the Small-Set Vertex Expansion problem that, given a graph $G = (V, E)$ of maximum degree $d_{\max}$ and parameters $\varepsilon \in (0,1)$ and $\delta \in (0,1/2)$, finds a set $S \subset V$ of size at most $(1+\varepsilon)\delta n$ such that*

$$\phi^V(S) \le O_\varepsilon\left(\sqrt{\phi^V_{G,\delta} \log d_{\max}} \cdot \delta^{-1} \log \delta^{-1} \log\log \delta^{-1} + \delta^{-1}\phi^V_{G,\delta}\right)$$
$$= \widetilde{O}_\varepsilon\left(\delta^{-1}\sqrt{\phi^V_{G,\delta} \log d_{\max}} + \delta^{-1}\phi^V_{G,\delta}\right).$$

We note that the Small-Set Vertex Expansion problem for $\delta = 1/2$ is just the Vertex Expansion problem. In that case, Theorem 1.7 gives the same approximation guarantee as the algorithm by Feige, Hajiaghayi, and Lee [12] (see also [1]); Theorem 1.8 gives the same approximation guarantee as the algorithm by Louis, Raghavendra, and Vempala [20]. To the best of our knowledge, Theorem 1.7 and Theorem 1.8 are the first non-trivial approximation algorithms for SSVE. We mention that Chuzhoy et al. [9] considered a bipartite variant of SSVE (which is somewhat similar but not equivalent to the problem we study in this paper) and proved a hardness result for it in the regime where δ is polynomially small.

Remark 1.9. We note that the notion of hypergraph expansion is not directly related to the notion of coboundary expansion of simplicial complexes [15, 14, 11]. Aside from an obvious distinction that the former applies to hypergraphs and the latter applies to simplicial complexes, these two types of expansion measure very different parameters of hypergraphs/simplicial complexes. For instance, consider an r-uniform hypergraph $H = (V, E)$ (with $r \ge 3$), in which every two hyperedges intersect in at most $r-2$ vertices, and the corresponding simplicial complex $X = \{\sigma \subset e : e \in E\}$ over $\mathbb{F}_2$.

Then, the hypergraph expansion of H may be more or less arbitrary, while the coboundary expansion of X in dimension r is equal to 1.

Techniques. Our general approach to solving H-SSE is similar to the approach of Bansal et al. [5]. We recall how the algorithm by Bansal et al. [5] for (graph) SSE works. The algorithm solves a semidefinite programming relaxation for SSE and gets an SDP solution. The SDP solution assigns a vector $\bar{u}$ to each vertex u. Then the algorithm generates *an orthogonal separator*. Informally, an orthogonal separator S with distortion D is a random subset of vertices such that the following hold.

(a) If $\bar{u}$ and $\bar{v}$ are close to each other then the probability that u and v are separated by S is small; namely, it is at most $\alpha D\|\bar{u}-\bar{v}\|^2$, where α is a normalization factor such that $\Pr(u \in S) = \alpha\|\bar{u}\|^2$.

(b) If the angle between $\bar{u}$ and $\bar{v}$ is larger than a certain threshold, then the probability that both u and v are in S is much smaller than the probability that one of them is in S.

Bansal et al. [5] showed that condition (b) together with SDP constraints implies that S is of size at most $(1+\varepsilon)\delta n$ with sufficiently high probability. Then condition (a) implies that the expected number of cut edges is at most D times the SDP value. That means that S is a D–approximate solution to SSE.

If we run this algorithm on an instance of H-SSE, we will still find a set of size at most $(1+\varepsilon)\delta n$, but the cost of the solution might be very high. Indeed, consider a hyperedge e. Even though every two vertices u and v in e are *unlikely* to be separated by S, at least one pair out of $\binom{|e|}{2}$ pairs of vertices is quite likely to be separated by S; hence, e is quite likely to be cut by S. To deal with this problem, we develop *hypergraph orthogonal separators*. In the definition of a hypergraph orthogonal separator, we strengthen condition (a) by requiring that a hyperedge e is cut by S with small probability if all vertices in e are close to each other. Specifically, we require that

$$\Pr(e \text{ is cut by } S) \leq \alpha D \max_{u,v\in e} \|\bar{u}-\bar{v}\|^2 . \tag{1.1}$$

We show that there is a hypergraph orthogonal separator with distortion proportional to $\sqrt{\log n}$ (the distortion also depends on parameters of the orthogonal separator). Plugging this hypergraph orthogonal separator in the algorithm by Bansal et al. [5], we get Theorem 1.3. We also develop another variant of hypergraph orthogonal separators, ℓ_2–ℓ_2^2 orthogonal separators. An ℓ_2–ℓ_2^2 orthogonal separator with ℓ_2–distortion $D_{\ell_2}(r)$ and ℓ_2^2–distortion $D_{\ell_2^2}$ satisfies the following condition.[1]

$$\Pr(e \text{ is cut by } S) \leq \alpha D_{\ell_2}(|e|) \cdot \min_{w\in E} \|\bar{w}\| \cdot \max_{u,v\in e} \|\bar{u}-\bar{v}\| + \alpha D_{\ell_2^2} \cdot \max_{u,v\in e} \|\bar{u}-\bar{v}\|^2 . \tag{1.2}$$

We show that there is an ℓ_2-ℓ_2^2 hypergraph orthogonal separator whose ℓ_2 and ℓ_2^2 distortions do not depend on n. (In contrast, there is no hypergraph orthogonal separator whose distortion does not depend on n.) This result yields Theorem 1.4.

We now give a brief conceptual overview of our construction of hypergraph orthogonal separators. We use the framework developed by Chlamtáč et al. [8, Section 4.3] for (graph) orthogonal separators. For simplicity, we ignore vector normalization steps in this overview; we do not explain how we take into account vector lengths. Note, however, that these normalization steps are crucial. We first design a procedure that partitions the hypergraph into two pieces (the procedure labels every vertex with either 0 or 1). In a sense, each set S in the partition is a "very weak" hypergraph orthogonal separator. It satisfies property (1.1) with $D_0 \sim \sqrt{\log n}\log\log(1/\delta)$ and $\alpha_0 = 1/2$ and a weak variant of property (b): if the angle between vectors $\bar{u}$ and $\bar{v}$ is larger than the threshold then events $u \in S$ and $v \in S$ are "almost" independent. We repeat the procedure $l = \log_2(1/\delta) + O(1)$ times and obtain a partition of graph into $2^l = O(1/\delta)$ pieces. Then we randomly choose one set S among them; this set S is our hypergraph orthogonal separator. Note that by running the procedure many times we decrease exponentially in l the probability that two vertices, as in condition (b), belong to S. So condition (b) holds for S. Also, we effect the distortion in (1.1) in two ways. First, the probability that the edge is cut increases by a factor of l. That is, we get

$$\Pr(e \text{ is cut by } S) \leq l \times \alpha_0 D_0 \max_{u,v\in e} \|\bar{u}-\bar{v}\|^2 .$$

[1] It may look strange that we have two terms in the bound. One may expect that we can either have only term

$$D_{\ell_2^2} \max_{u,v\in e} \|\bar{u}-\bar{v}\|^2$$

(as in the previous definition) or only term

$$D_{\ell_2}(|e|) \cdot \min_{w\in E} \|\bar{w}\| \cdot \max_{u,v\in e} \|\bar{u}-\bar{v}\| .$$

However, the latter is not possible—there is no ℓ_2–ℓ_2^2 separator with $D_{\ell_2^2} = 0$.

$$\text{minimize} \sum_{e\in E} \max_{u,v\in e} \|\bar{u}-\bar{v}\|^2 \tag{2.1}$$

subject to:

$$\sum_{v\in V} \langle \bar{u},\bar{v}\rangle \le \delta n \cdot \|\bar{u}\|^2 \qquad \text{for every } u \in V\,, \tag{2.2}$$

$$\sum_{u\in V} \|\bar{u}\|^2 = 1\,, \tag{2.3}$$

$$\|\bar{u}-\bar{v}\|^2 + \|\bar{v}-\bar{w}\|^2 \ge \|\bar{u}-\bar{w}\|^2 \qquad \text{for every } u,v,w \in V\,, \tag{2.4}$$

$$0 \le \langle \bar{u},\bar{v}\rangle \le \|\bar{u}\|^2 \qquad \text{for every } u,v \in V\,. \tag{2.5}$$

Figure 1: SDP relaxation for H-SSE.

Second, the probability that we choose a vertex u goes down from $\|\bar{u}\|^2/2$ to $\Omega(\delta)\|\bar{u}\|^2$ since roughly speaking we choose one set S among $O(1/\delta)$ possible sets. That is, the parameter α of S is $\Omega(\delta)$. Therefore,

$$\Pr(e \text{ is cut by } S) \le \alpha(\alpha_0 l D_0/\alpha) \max_{u,v\in e} \|\bar{u}-\bar{v}\|^2\,.$$

That is, we get a hypergraph orthogonal separator with distortion $(\alpha_0 l D_0/\alpha) \sim \widetilde{O}(\delta^{-1}\sqrt{\log n})$. The construction of ℓ_2–ℓ_2^2 orthogonal separators is similar but a bit more technical.

Organization. We present our SDP relaxation and introduce our main technique, hypergraph orthogonal separators, in Section 2. We describe our first algorithm for H-SSE in Section 3, and then describe an algorithm that generates hypergraph orthogonal separators in Section 4. We define ℓ_2–ℓ_2^2 hypergraph orthogonal separators, give an algorithm that generates them, and then present our second algorithm for H-SSE in Sections 5 and 6. Finally, we show a simple SDP integrality gap for H-SSE in Section 7. This integrality gap also gives a lower bound on the quality of m-orthogonal separators. We give a proof of Theorem 8.2 in Section 8.

2 Preliminaries

2.1 SDP relaxation for Hypergraph Small-Set Expansion

We use the SDP relaxation for H-SSE shown in Figure 1. There is an SDP variable $\bar{u}$ for every vertex $u \in V$. Our spreading constraint (2.2) is a standard spreading constraint used to ensure that only a small number of vectors have "large" norms [23, 5, 17]. An illustrative example is the SDP solution obtained by fixing a set $S \subset V$, and defining $\bar{u} = a/\sqrt{|S|}$, if $u \in S$; $\bar{u} = 0$, otherwise, where a is a fixed unit vector. This solution will satisfy the spreading constraint (2.2) only if $|S| \le \delta n$.

Every combinatorial solution S (with $|S| \le \delta n$) defines the corresponding (intended) SDP solution: $\bar{u} = a/\sqrt{|S|}$, if $u \in S$; $\bar{u} = 0$, otherwise, where a is a fixed unit vector. It is easy to see that this solution satisfies all SDP constraints. Note that $\max_{u,v\in e} \|\bar{u}-\bar{v}\|^2$ is equal to $1/|S|$, if e is cut, and to 0, otherwise.

Therefore, the objective function equals

$$\sum_{e\in E} \max_{u,v\in e} \|\bar{u}-\bar{v}\|^2 = \sum_{e\in E_{\mathrm{cut}}(S)} \frac{1}{|S|} = \frac{E_{\mathrm{cut}}(S)}{S} = \phi(S)\,.$$

Thus our SDP for H-SSE is indeed a relaxation. We denote the value of the optimal SDP solution by $\mathsf{sdpcost}$.

2.2 Hypergraph orthogonal separators

The main technical tool for proving Theorem 1.3 is *hypergraph orthogonal separators*. *Orthogonal separators* were introduced by Chlamtáč, Makarychev, and Makarychev [8] (see also [5, 17, 21]) and were previously used for solving Unique Games and various graph partitioning problems. In this paper, we extend the technique of orthogonal separators to hypergraphs and introduce hypergraph orthogonal separators. We then use hypergraph orthogonal separators to solve H-SSE. In Section 5, we introduce another version of hypergraph orthogonal separators, ℓ_2–ℓ_2^2 hypergraph orthogonal separators, and then use them to prove Theorems 1.4 and 6.3.

Definition 2.1 (Hypergraph Orthogonal Separators)**.** Let $\{\bar{u} : u \in V\}$ be a set of vectors in the unit ball that satisfy ℓ_2^2–triangle inequalities (2.4) and (2.5). We say that a random set $S \subset V$ is *a hypergraph m-orthogonal separator* with distortion $D \geq 1$, probability scale $\alpha > 0$, and separation threshold $\beta \in (0,1)$ if it satisfies the following properties.

1. For every $u \in V$, $\Pr(u \in S) = \alpha\|\bar{u}\|^2$.
2. For every u and v such that $\|\bar{u}-\bar{v}\|^2 \geq \beta \min(\|\bar{u}\|^2, \|\bar{v}\|^2)$,
$$\Pr\left(u \in S \text{ and } v \in S\right) \leq \alpha \frac{\min(\|\bar{u}\|^2, \|\bar{v}\|^2)}{m}\,.$$
3. For every $e \subset V$, $\Pr\left(e \text{ is cut by } S\right) \leq \alpha D \max_{u,v\in e} \|\bar{u}-\bar{v}\|^2$.

The definition of a hypergraph m-orthogonal separator is similar to that of a (graph) m-orthogonal separator: a random set S is an m-orthogonal separator if it satisfies properties 1, 2, and property $3'$, which is property 3 restricted to edges e of size 2.

$3'$. For every (u,v), $\Pr\left(e \text{ is cut by } S\right) \leq \alpha D\|\bar{u}-\bar{v}\|^2$.

In this paper, we design an algorithm that generates a hypergraph m-orthogonal separator with distortion $O_\beta(\sqrt{\log n}\cdot m \log m \log\log m)$. We note that the distortion of *any* hypergraph orthogonal separator must depend on m at least linearly (see Section 7). We remark that there are two constructions of (graph) orthogonal separators, "orthogonal separators via ℓ_1" and "orthogonal separators via ℓ_2," with distortions, $O_\beta(\sqrt{\log n}\log m)$ and $O_\beta(\sqrt{\log n \log m})$, respectively (presented in [8]). Our construction of hypergraph orthogonal separators uses the framework of orthogonal separators via ℓ_1. We prove the following theorem in Section 4.

Theorem 2.2. *There is a polynomial-time randomized algorithm that given a set of vertices V, a set of vectors $\{\bar{u}\}$ satisfying ℓ_2^2–triangle inequalities (2.4) and (2.5), parameters $m \geq 2$ and $\beta \in (0,1)$, generates a hypergraph m-orthogonal separator with probability scale $\alpha \geq 1/n$ and distortion $D = O(\beta^{-1} m \log m \log\log m \times \sqrt{\log n})$.*

3 Algorithm for Hypergraph Small-Set Expansion

In this section, we present our algorithm for Hypergraph Small-Set Expansion. Our algorithm uses hypergraph orthogonal separators that we describe in Section 4. We use the approach of Bansal et al. [5]. Suppose that we are given a polynomial-time algorithm that generates hypergraph m-orthogonal separators with distortion $D(m,\beta)$ (with probability scale $\alpha > 1/\mathrm{poly}(n)$). We show how to get a $D^* = 4D(4/(\varepsilon\delta), \varepsilon/4)$–approximation for H-SSE.

Theorem 3.1. *There is a randomized polynomial-time approximation algorithm for the Hypergraph Small-Set Expansion problem that given a hypergraph $H = (V,E)$, and parameters $\varepsilon \in (0,1)$ and $\delta \in (0,1/2)$ finds a set $S \subset V$ of size at most $(1+\varepsilon)\delta n$ such that $\phi(S) \leq 4D(4/(\varepsilon\delta), \varepsilon/4) \cdot \phi^*_{H,\delta}$.*

Proof. We solve the SDP relaxation for H-SSE and obtain an SDP solution $\{\bar{u}\}$. Consider a hypergraph orthogonal separator S with $m = 4/(\varepsilon\delta)$ and $\beta = \varepsilon/4$. Define a set S':

$$S' = \begin{cases} S & \text{if } |S| \leq (1+\varepsilon)\delta n, \\ \varnothing & \text{otherwise.} \end{cases}$$

Clearly, $|S'| \leq (1+\varepsilon)\delta n$. Bansal et al. [5] showed the following lemma (see also Theorem A.1 in [21]); for completeness, we reproduce their proof here.

Lemma 3.2 (Bansal et al. [5])**.** $\Pr(u \in S') \in \left[(\alpha/2)\,\|\bar{u}\|^2, \alpha\|\bar{u}\|^2\right]$ *for every $u \in V$.*

Proof. Clearly, $\Pr(u \in S') \leq \Pr(u \in S) = \alpha\|\bar{u}\|^2$. Now, let

$$A_u := \left\{\bar{v} : \|\bar{u} - \bar{v}\|^2 \geq \beta\|\bar{u}\|^2\right\} \qquad \text{and} \qquad B_u := \left\{\bar{v} : \|\bar{u} - \bar{v}\|^2 < \beta\|\bar{u}\|^2\right\}.$$

We show that only a small fraction of A_u belongs to S, and that the set B_u is small.

For any $v \in B_u$, we have

$$\|\bar{u}\|^2 - \langle \bar{u}, \bar{v} \rangle = \|\bar{u} - \bar{v}\|^2 - (\|\bar{v}\|^2 - \langle \bar{u}, \bar{v} \rangle) < \beta\|\bar{u}\|^2.$$

Therefore, $\langle \bar{u}, \bar{v} \rangle > (1-\beta)\|\bar{u}\|^2$. Now,

$$|B_u| \leq \sum_{v \in B_u} \frac{\langle \bar{u}, \bar{v} \rangle}{(1-\beta)\|\bar{u}\|^2} \leq \frac{1}{(1-\beta)\|\bar{u}\|^2} \sum_{v \in V} \langle \bar{u}, \bar{v} \rangle \leq \frac{\delta n}{1-\beta} \leq \delta(1+2\beta)n = \big(1+\varepsilon/2\big)\delta n.$$

Here, the last inequality uses (2.2).

For an arbitrary $v \in A_u$, by the first and second properties of orthogonal separators, $\Pr(v \in S \mid u \in S) \leq 1/m$. Thus, $\mathbb{E}[|A_u \cap S| \mid u \in S] \leq n/m$. Therefore, using Markov's inequality,

$$\Pr(|A_u \cap S| \geq \varepsilon\delta n/2 \mid u \in S) \leq \frac{n/m}{\varepsilon\delta n/2} \leq \frac{1}{2}.$$

Therefore,

$$\Pr(|S| \leq (1+\varepsilon)\delta n) \geq \Pr(|A_u \cap S| \leq \varepsilon\delta n/2 \mid u \in S) \geq \frac{1}{2}$$

and

$$\Pr\left(u \in S'\right) \geq \alpha\|\bar{u}\|^2 \Pr(|S| \leq (1+\varepsilon)\delta n) \geq \frac{\alpha\|\bar{u}\|^2}{2}. \qquad \square$$

Note that

$$\Pr\left(S' \text{ cuts edge } e\right) \leq \Pr(S \text{ cuts edge } e) \leq \alpha D^* \max_{u,v\in e} \|\bar{u}-\bar{v}\|^2$$

where D^* denotes $D(4/(\varepsilon\delta), \varepsilon/4)$ for the sake of brevity. Let

$$Z = |S'| - \frac{|E_{\text{cut}}(S')|}{4D^* \cdot \mathsf{sdpcost}}.$$

We have

$$\begin{aligned}\mathbb{E}[Z] &= \mathbb{E}\left[|S'|\right] - \frac{\mathbb{E}[|E_{\text{cut}}(S')|]}{4D^* \cdot \mathsf{sdpcost}} \geq \sum_{u\in V} \frac{\alpha}{2} \cdot \|\bar{u}\|^2 - \frac{\sum_{e\in E} \alpha D^* \max_{u,v\in e} \|\bar{u}-\bar{v}\|^2}{4D^* \cdot \mathsf{sdpcost}} \\ &= \frac{\alpha}{2} - \frac{\alpha D^* \mathsf{sdpcost}}{4D^* \cdot \mathsf{sdpcost}} = \frac{\alpha}{4}.\end{aligned}$$

Since $Z \leq |S'| \leq (1+\varepsilon)\delta n < n$ (always), by Markov's inequality, we have $\Pr(Z > 0) \geq \alpha/(4n)$ and hence

$$\Pr\left(|E_{\text{cut}}(S')|/|S'| < 4D^* \cdot \mathsf{sdpcost}\right) \geq \alpha/(4n).$$

We sample S independently $4n/\alpha$ times and return the first set S' such that

$$\frac{|E_{\text{cut}}(S')|}{|S'|} < 4D^* \cdot \mathsf{sdpcost}.$$

This gives a set S' such that $|S'| \leq (1+\varepsilon)\delta n$, and $\phi(S') \leq 4D^* \phi^*_{H,\delta}$. The algorithm succeeds (finds such a set S') with a constant probability. By repeating the algorithm n times, we can make the success probability exponentially close to 1. $\square$

In Section 4, we describe how to generate an m-hypergraph orthogonal separator with distortion

$$D = O\left(\sqrt{\log n} \times \beta^{-1} m \log m \log\log m\right).$$

That gives us an algorithm for H-SSE with approximation factor

$$O_\varepsilon\left(\delta^{-1} \log \delta^{-1} \log\log \delta^{-1} \times \sqrt{\log n}\right).$$

4 Generating hypergraph orthogonal separators

In this section, we present an algorithm that generates a hypergraph m-orthogonal separator. At the high level, the algorithm is similar to the algorithm for generating orthogonal separators from Section 4.3 in the paper by Chlamtáč, Makarychev, and Makarychev [8]. We use a different procedure for generating words $W(u)$ (see below) and set parameters differently; also the analysis of our algorithm is different. As in [8], our algorithm has three main steps.

Our first step is to use a "normalization" map φ (the step is identical to one in [8]). This normalization transforms the set of vectors into a set of functions in $L_2[0,\infty]$, so that the image of every non-zero vector is a function with L_2 norm 1; the images of orthogonal vectors are orthogonal; the distance between the images of u and v is roughly preserved. Moreover, the images satisfy L_2^2 triangle inequalities. The map φ is defined as follows.

$$\varphi(u)(t) = \begin{cases} u, & \text{if } t \leq 1/\|\bar{u}\|^2, \\ 0, & \text{otherwise.} \end{cases}$$

Formally, φ has the following properties.

1. For all vertices u, v, w, $\|\varphi(\bar{u}) - \varphi(\bar{v})\|_2^2 + \|\varphi(\bar{v}) - \varphi(\bar{w})\|_2^2 \geq \|\varphi(\bar{u}) - \varphi(\bar{w})\|_2^2$.
2. For all nonzero vertices u and v, $\langle \varphi(\bar{u}), \varphi(\bar{v}) \rangle = \dfrac{\langle \bar{u}, \bar{v} \rangle}{\max(\|\bar{u}\|^2, \|\bar{v}\|^2)}$.
3. In particular, for every $\bar{u} \neq 0$, $\|\varphi(\bar{u})\|_2^2 = \langle \varphi(\bar{u}), \varphi(\bar{u}) \rangle = 1$. Also, $\varphi(0) = 0$.
4. For all non-zero vectors $\bar{u}$ and $\bar{v}$, $\|\varphi(\bar{u}) - \varphi(\bar{v})\|_2^2 \leq \dfrac{2\,\|\bar{u} - \bar{v}\|^2}{\max(\|\bar{u}\|^2, \|\bar{v}\|^2)}$.

In the second step of our algorithm, we also use the following theorem by Arora, Lee, and Naor [3] (see also [4]).

Theorem 4.1 (Arora, Lee, and Naor (2005), Theorem 3.1)**.** *There exist constants $C \geq 1$ and $p \in (0, 1/4)$ such that for every n unit vectors x_u ($u \in V$), satisfying ℓ_2^2–triangle inequalities (2.4), and every $\Delta > 0$, the following holds. There exists a random subset U of V such that for every $u, v \in V$ with $\|x_u - x_v\|^2 \geq \Delta$,*

$$\Pr\left(u \in U \text{ and } d(v, U) \geq \frac{\Delta}{C\sqrt{\log n}} \right) \geq p\,,$$

where $d(v, U) = \min_{u \in U} \|x_u - x_v\|^2$.

The second step of our algorithm is summarised by the following lemma. In this step, we describe an algorithm that randomly assigns each vertex u a symbol, either 0 or 1. In this step, we deviate from the algorithm in [8].

Lemma 4.2. *There is a randomized polynomial-time algorithm that, given a finite set V, unit vectors $\varphi(\bar{u})$ for $u \in V$ satisfying ℓ_2^2-triangle inequalities, and a parameter $\beta \in (0, 1)$, returns a random assignment $\omega : V \to \{0, 1\}$ that satisfies the following properties.*

- *For every u and v such that $\|\varphi(\bar{u})-\varphi(\bar{v})\|^2 \geq \beta$, we have* $\Pr(\omega(u)\neq\omega(v)) \geq 2p$, *where $p>0$ is the constant from Theorem 4.1.*
- *For every set $e\subset V$ of size at least 2,*

$$\Pr(\omega(u)\neq\omega(v) \text{ for some } u,v\in e) \leq O\big(\beta^{-1}\sqrt{\log n}\max_{u,v\in e}\|\varphi(\bar{u})-\varphi(\bar{v})\|^2\big).$$

Proof. Let U be the random set from Theorem 4.1 for vectors $x_u=\varphi(\bar{u})$ and $\Delta=\beta$. Choose $t\in(0,\Delta/(C\sqrt{\log n}))$ uniformly at random. Let

$$\omega(u)=\begin{cases}0, & \text{if } d(u,U)\leq t,\\ 1, & \text{otherwise.}\end{cases}$$

Consider first vertices u and v such that $\|\varphi(\bar{u})-\varphi(\bar{v})\|^2\geq\beta$. By Theorem 4.1,

$$\Pr\left(u\in U \text{ and } d(v,U)\geq\frac{\Delta}{C\sqrt{\log n}}\right)\geq p \qquad \text{and} \qquad \Pr\left(v\in U \text{ and } d(u,U)\geq\frac{\Delta}{C\sqrt{\log n}}\right)\geq p.$$

Note that in the former case, when $u\in U$ and $d(v,U)\geq\Delta/(C\sqrt{\log n})$, we have $\omega(u)=0$ and $\omega(v)=1$; in the latter case, when $v\in U$ and $d(u,U)\geq\Delta/(C\sqrt{\log n})$, we have $\omega(v)=0$ and $\omega(u)=1$. Therefore, the probability that $\omega(u)\neq\omega(v)$ is at least $2p$.

Now consider a set $e\subset V$ of size at least 2. Let

$$\tau_m=\min_{w\in e} d(U,\varphi(\bar{w})) \qquad \text{and} \qquad \tau_M=\max_{w\in e} d(U,\varphi(\bar{w})).$$

We have $\tau_M-\tau_m\leq\max_{u,v\in e}\|\varphi(\bar{u})-\varphi(\bar{v})\|^2$. Note that if $t<\tau_m$ then $\omega(u)=1$ for all $u\in e$; if $t\geq\tau_M$ then $\omega(u)=0$ for all $u\in e$. Thus $\omega(u)\neq\omega(v)$ for some $u,v\in e$ only if $t\in[\tau_m,\tau_M)$. Since the probability density of the random variable t is at most $C\sqrt{\log n}$, we get,

$$\Pr(\exists u,v\in e:\omega(u)\neq\omega(v))\leq\Pr(t\in[\tau_m,\tau_M))\leq\frac{C\sqrt{\log n}}{\beta}\max_{u,v\in e}\|\varphi(\bar{u})-\varphi(\bar{v})\|^2. \qquad \square$$

We now amplify the result of Lemma 4.2.

Lemma 4.3. *There is a randomized polynomial time algorithm that given V, vectors $\varphi(\bar{u})$ and $\beta\in(0,1)$ as in Lemma 4.2, and a parameter $m\geq 2$, returns a random assignment $\tilde{\omega}:V\to\{0,1\}$ such that the following hold.*

- *For every u and v such that $\|\varphi(\bar{u})-\varphi(\bar{v})\|^2\geq\beta$,*

$$\Pr(\tilde{\omega}(u)\neq\tilde{\omega}(v))\geq 1/2-1/\log_2 m.$$

- *For every set $e\subset V$ of size at least 2,*

$$\Pr(\tilde{\omega}(u)\neq\tilde{\omega}(v) \text{ for some } u,v\in e)\leq O\big(\beta^{-1}\sqrt{\log n}\cdot\log\log m\cdot\max_{u,v\in e}\|\varphi(\bar{u})-\varphi(\bar{v})\|^2\big).$$

Proof. Let
$$K = \max\left(\left\lceil \frac{\log_2\log_2 m}{-\log_2(1-4p)}\right\rceil, 1\right).$$
We independently sample K assignments $\omega_1, \dots, \omega_K$ as described in Lemma 4.2. Let
$$\tilde{\omega}(u) = \omega_1(u) \oplus \cdots \oplus \omega_K(u),$$
where $\oplus$ denotes addition modulo 2. Consider u and v such that $\|\varphi(\bar{u}) - \varphi(\bar{v})\|^2 \geq \beta$. Let
$$\tilde{p} = \Pr(\omega_i(u) \neq \omega_i(v)) \geq 2p \quad \text{for} \quad i \in \{1, \dots, K\}$$
(the expression does not depend on the value of i since all ω_i are identically distributed). Note that $\tilde{\omega}(u) \neq \tilde{\omega}(v)$ if and only if $\omega_i(u) \neq \omega_i(v)$ for an odd number of indices i. Therefore,
$$\begin{aligned}\Pr(\omega(u) \neq \omega(v)) &= \sum_{0 \leq k < K/2} \binom{K}{2k+1} \tilde{p}^{2k+1}(1-\tilde{p})^{K-2k-1} = \frac{1-(1-2\tilde{p})^K}{2} \\ &\geq \frac{1-(1-4p)^K}{2} \geq \frac{1}{2} - \frac{1}{\log_2 m}.\end{aligned}$$
Now let $e \subset V$ be a subset of size at least 2. We have, by the union bound,
$$\Pr(\tilde{\omega}(u) \neq \tilde{\omega}(v)) \leq \Pr(\omega_i(u) \neq \omega_i(v) \text{ for some } i) \leq O\big(K\beta^{-1}\sqrt{\log n}\max_{u,v \in e}\|\varphi(\bar{u}) - \varphi(\bar{v})\|^2\big). \qquad \square$$

In the third step (step 3 of Algorithm 1), we generate "words" using the $\tilde{\omega}(\cdot)$. This step is again similar to one of the steps in the paper [8]. We present our complete algorithm for the hypergraph orthogonal separator below, and we analyze our algorithm in Theorem 4.4.

Algorithm 1.

1. Set $l = \lceil \log_2 m/(1-\log_2(1+2/\log_2 m))\rceil = \log_2 m + O(1)$.
2. Sample l independent assignments $\tilde{\omega}_1, \dots, \tilde{\omega}_l$ using Lemma 4.3.
3. For every vertex u, define word $W(u) = (\tilde{\omega}_1(u), \dots, \tilde{\omega}_l(u)) \in \{0,1\}^l$.
4. If $n \geq 2^l$, pick a word $W \in \{0,1\}^l$ uniformly at random. If $n < 2^l$, pick a random word $W \in \{0,1\}^l$ so that $\Pr_W(W = W(u)) = 1/n$ for every $u \in V$. This is possible since the number of distinct words constructed in step 3 is at most n (we may pick a word W not equal to any $W(u)$).
5. Pick $r \in (0,1)$ uniformly at random.
6. Let $S = \big\{u \in V : \|\bar{u}\|^2 \geq r \text{ and } W(u) = W\big\}$.

Theorem 4.4. *Random set S is a hypergraph m-orthogonal separator with distortion*
$$D = O\left(\sqrt{\log n} \times \frac{m \log m \log\log m}{\beta}\right),$$
probability scale $\alpha \geq 1/n$ and separation threshold β.

Proof. We verify that S satisfies properties 1–3 in the definition of a hypergraph m-orthogonal separator with $\alpha = \max(1/2^l, 1/n)$.

Property 1. Let $\alpha = \max(1/2^l, 1/n)$. We compute the probability that $u \in S$. Observe that $u \in S$ if and only if $W(u) = W$ and $r \le \|\bar{u}\|^2$ (these two events are independent). If $n \ge 2^l$, the probability that $W = W(u)$ is $1/2^l$ since we choose W uniformly at random from $\{0,1\}^l$; if $n < 2^l$ the probability is $1/n$. That is, $\Pr(W = W(u)) = \max(1/2^l, 1/n) = \alpha$. The probability that $r \le \|\bar{u}\|^2$ is $\|\bar{u}\|^2$. We conclude that property 1 holds.

Property 2. Consider two vertices u and v such that $\|\bar{u} - \bar{v}\|^2 \ge \beta \min(\|\bar{u}\|^2, \|\bar{v}\|^2)$. Assume without loss of generality that $\|\bar{u}\|^2 \le \|\bar{v}\|^2$. Note that $u, v \in S$ if and only if $r \le \|\bar{u}\|^2$ and $W = W(u) = W(v)$. We first upper bound the probability that $W(u) = W(v)$. We have

$$2\langle \bar{u}, \bar{v}\rangle = \|\bar{u}\|^2 + \|\bar{v}\|^2 - \|\bar{u} - \bar{v}\|^2 \le (1-\beta)\|\bar{u}\|^2 + \|\bar{v}\|^2 \le (2-\beta)\|\bar{v}\|^2 .$$

Therefore, $2\langle \bar{u}, \bar{v}\rangle / \|\bar{v}\|^2 \le 2 - \beta$. Hence,

$$\|\varphi(\bar{u}) - \varphi(\bar{v})\|^2 = 2 - 2\langle \varphi(\bar{u}), \varphi(\bar{v})\rangle = 2 - \frac{2\langle \bar{u}, \bar{v}\rangle}{\max(\|\bar{u}\|^2, \|\bar{v}\|^2)} \ge \beta .$$

From Lemma 4.3 we get that $\Pr(\tilde{\omega}_i(u) \ne \tilde{\omega}_i(v)) \ge 1/2 - 1/\log_2 m$ for every i. The probability that $W(u) = W(v)$ is at most

$$\left(\frac{1}{2} + \frac{1}{\log_2 m}\right)^l \le 1/m .$$

Therefore we have, as required,

$$\begin{aligned}\Pr(u \in S, v \in S) = \Pr\left(r \le \min(\|\bar{u}\|^2, \|\bar{v}\|^2)\right) &\times \Pr(W = W(u) = W(v) \mid W(u) = W(v)) \\ &\times \Pr(W(u) = W(v)) \le \min(\|\bar{u}\|^2, \|\bar{v}\|^2) \times \alpha \times (1/m) .\end{aligned}$$

Property 3. Let e be an arbitrary subset of V, $|e| \ge 2$. Let $\rho_m = \min_{w \in e} \|\bar{w}\|^2$ and $\rho_M = \max_{w \in e} \|\bar{w}\|^2$. Note that

$$\rho_M - \rho_m = \|\bar{w}_1\|^2 - \|\bar{w}_2\|^2 \le \|\bar{w}_1 - \bar{w}_2\|^2 \le \max_{u,v \in e} \|\bar{u} - \bar{v}\|^2 ,$$

for some $w_1, w_2 \in e$. Here we used that SDP constraint (2.5) implies that $\|\bar{w}_1\|^2 - \|\bar{w}_2\|^2 \le \|\bar{w}_1 - \bar{w}_2\|^2$.

Let $A = \left\{u \in e : \|\bar{u}\|^2 \ge r\right\}$. We have $S \cap e = \{u \in A : W(u) = W\}$. Therefore, if e is cut by S then one of the following events happens.

- Event $\mathcal{E}_1$: $A \ne e$ and $S \cap e \ne \varnothing$.
- Event $\mathcal{E}_2$: $A = e$ and $A \cap S \ne \varnothing$, $A \cap S \ne A$.

If $\mathcal{E}_1$ happens then $r \in [\rho_m, \rho_M]$ since $A \ne e$ and $A \ne \varnothing$ (because $A \supset S \cap e \ne \varnothing$). We have

$$\Pr(\mathcal{E}_1) \le \Pr(r \in (\rho_m, \rho_M]) \le |\rho_M - \rho_m| \le \max_{u,v \in e} \|\bar{u} - \bar{v}\|^2.$$

If $\mathcal{E}_2$ happens then (1) $r \le \rho_m$ (since $A = e$) and (2) $W(u) \ne W(v)$ for some $u, v \in e$. The probability that $r \le \rho_m$ is ρ_m. We now upper bound the probability that $W(u) \ne W(v)$ for some $u, v \in e$. For each $i \in \{1, \dots, l\}$,

$$\begin{aligned}\Pr(\tilde{\omega}_i(u) \ne \tilde{\omega}_i(v) \text{ for some } u,v \in e) &\le O\big(\beta^{-1}\sqrt{\log n}\cdot \log\log m\big) \max_{u,v\in e} \|\varphi(\bar{u}) - \varphi(\bar{v})\|^2 \\ &\le O\big(\beta^{-1}\sqrt{\log n}\cdot \log\log m\big) \max_{u,v\in e} \frac{2\|\bar{u}-\bar{v}\|^2}{\min(\|\bar{u}\|^2, \|\bar{v}\|^2)} \\ &\le O\big(\beta^{-1}\sqrt{\log n}\cdot \log\log m\big) \times \rho_m^{-1} \times \max_{u,v\in e} \|\bar{u}-\bar{v}\|^2.\end{aligned}$$

By the union bound over $i \in \{1, \dots, l\}$, the probability that $W(u) \ne W(v)$ for some $u, v \in e$ is at most

$$O\big(l \times \beta^{-1}\sqrt{\log n}\cdot \log\log m\big) \times \rho_m^{-1} \times \max_{u,v\in e} \|\bar{u}-\bar{v}\|^2.$$

Therefore,

$$\begin{aligned}\Pr(\mathcal{E}_2) &\le \rho_m \times O\big(l \times \beta^{-1}\sqrt{\log n}\, \log\log m\big) \times \rho_m^{-1} \times \max_{u,v\in e} \|\bar{u}-\bar{v}\|^2 \\ &\le O\big(\beta^{-1}\sqrt{\log n}\, \log m \log\log m\big) \times \max_{u,v\in e} \|\bar{u}-\bar{v}\|^2.\end{aligned}$$

We get that the probability that e is cut by S is at most

$$\Pr(\mathcal{E}_1) + \Pr(\mathcal{E}_2) \le O\big(\beta^{-1}\sqrt{\log n}\, \log m \log\log m\big) \times \max_{u,v\in e} \|\bar{u}-\bar{v}\|^2.$$

For $D = O(\beta^{-1}\sqrt{\log n}\, \log m \log\log m)/\alpha$, we get $\Pr(e \text{ is cut by } S) \le \alpha D \max_{u,v\in e} \|\bar{u}-\bar{v}\|^2$. Note that $\alpha \ge 1/2^l \ge \Omega(1/m)$. Thus $D \le O(\beta^{-1}\sqrt{\log n}\, m \log m \log\log m)$. □

5 ℓ_2–ℓ_2^2 hypergraph orthogonal separators

In this section, we present another variant of hypergraph orthogonal separators, which we call ℓ_2–ℓ_2^2 hypergraph orthogonal separators. The advantage of ℓ_2–ℓ_2^2 hypergraph orthogonal separators is that their distortions do not depend on n (the number of vertices). Then in Section 6, we use ℓ_2–ℓ_2^2 hypergraph orthogonal separators to prove Theorem 6.3 (which, in turn, implies Theorem 1.4).

Definition 5.1 (ℓ_2–ℓ_2^2 Hypergraph Orthogonal Separator)**.** Let $\{\bar{u} : u \in V\}$ be a set of vectors in the unit ball. We say that a random set $S \subset V$ is *an ℓ_2–ℓ_2^2 hypergraph m-orthogonal separator* with ℓ_2–distortion $D_{\ell_2} : \mathbb{N} \to \mathbb{R}$, ℓ_2^2–distortion $D_{\ell_2^2}$, probability scale $\alpha > 0$, and separation threshold $\beta \in (0,1)$ if it satisfies the following properties.

1. For every $u \in V$, $\Pr(u \in S) = \alpha\|\bar{u}\|^2$.

2. For every u and v such that $\|\bar{u}-\bar{v}\|^2 \ge \beta \min(\|\bar{u}\|^2, \|\bar{v}\|^2)$,

$$\Pr(u \in S \text{ and } v \in S) \le \alpha \frac{\min(\|\bar{u}\|^2, \|\bar{v}\|^2)}{m}.$$

3. For every $e \subset V$,

$$\Pr\left(e \text{ is cut by } S\right) \leq \alpha D_{\ell_2^2} \cdot \max_{u,v\in e} \|\bar{u}-\bar{v}\|^2 + \alpha D_{\ell_2}(|e|) \cdot \min_{w\in e} \|\bar{w}\| \cdot \max_{u,v\in e} \|\bar{u}-\bar{v}\| .$$

(This definition differs from Definition 2.1 only in item 3.)

Theorem 5.2. *There is a polynomial-time randomized algorithm that given a set of vertices V, a set of vectors $\{\bar{u}\}$ and parameters m and β generates an ℓ_2–ℓ_2^2 hypergraph m-orthogonal separator with probability scale $\alpha \geq 1/n$ and distortions:*

$$D_{\ell_2^2} = O(m) \qquad \text{and} \qquad D_{\ell_2}(r) = O(\beta^{-1/2}\sqrt{\log r}\, m \log m \log\log m).$$

Note that distortions $D_{\ell_2^2}$ and D_{ℓ_2} do not depend on n.

The algorithm and its analysis are very similar to those in the proof of Theorem 2.2. The only difference is that we use another procedure to generate random assignments $\omega : V \to \{0,1\}$. The following lemma is an analog of Lemma 4.2.

Lemma 5.3. *There is a randomized polynomial time algorithm that given a finite set V, vectors $\varphi(\bar{u})$ for $u \in V$, and a parameter $\beta \in (0,1)$, returns a random assignment $\omega : V \to \{0,1\}$ that satisfies the following properties.*

- *For every set $e \subset V$ of size at least 2,*

$$\Pr\left(\omega(u) \neq \omega(v) \text{ for some } u,v \in e\right) \leq O\left(\beta^{-1/2}\sqrt{\log|e|}\right) \times \max_{u,v\in e} \|\varphi(\bar{u}) - \varphi(\bar{v})\| .$$

- *For every u and v such that $\|\varphi(\bar{u}) - \varphi(\bar{v})\|^2 \geq \beta$,*

$$\Pr\left(\omega(u) \neq \omega(v)\right) \geq 0.3 .$$

Proof. We sample a random Gaussian vector $g \sim \mathcal{N}(0, I_n)$ (each component g_i of g is distributed as $\mathcal{N}(0,1)$, all random variables g_i are independent). Let N be a Poisson process on $\mathbb{R}$ with rate $1/\sqrt{\beta}$. Let $w(u) = 1$ if $N(\langle g, \varphi(\bar{u})\rangle)$ is even, and $w(u) = 0$ if $N(\langle g, \varphi(\bar{u})\rangle)$ is odd. Note that $\omega(u) = \omega(v)$ if and only if $N(\langle g, \varphi(\bar{u})\rangle) - N(\langle g, \varphi(\bar{v})\rangle)$ is even.

Consider a set $e \subset V$ of size at least 2. Denote $\operatorname{diam}(e) = \max_{u,v\in e} \|\varphi(\bar{u}) - \varphi(\bar{v})\|$. Let

$$\tau_m = \min_{w\in e} \langle g, \varphi(\bar{w})\rangle \qquad \text{and} \qquad \tau_M = \max_{w\in e} \langle g, \varphi(\bar{w})\rangle .$$

Note that

$$N(\tau_m) = \min_{w\in e} N(\langle g, \varphi(\bar{w})\rangle) \qquad \text{and} \qquad N(\tau_M) = \max_{w\in e} N(\langle g, \varphi(\bar{w})\rangle) .$$

If all numbers $N(\langle g, \varphi(\bar{u})\rangle)$ (for $u \in e$) are equal then $\omega(u) = \omega(v)$ for all $u, v \in e$. Thus if $\omega(u) \neq \omega(v)$ for some $u, v \in e$ then $N(\langle g, \varphi(\bar{u})\rangle) \neq N(\langle g, \varphi(\bar{v})\rangle)$ for some $u, v \in e$. In particular, then $N(\tau_M) - N(\tau_m) > 0$. Given g, $N(\tau_M) - N(\tau_m)$ is a Poisson random variable with rate $(\tau_M - \tau_m)/\sqrt{\beta}$. We have

$$\begin{aligned}\Pr\left(\omega(u) \neq \omega(v) \text{ for some } u,v \in e \mid g\right) &\leq \Pr\left(N(\tau_M) - N(\tau_m) > 0 \mid g\right) \\ &= 1 - e^{-(\tau_M - \tau_m)/\sqrt{\beta}} \leq \beta^{-1/2}(\tau_M - \tau_m) .\end{aligned}$$

Let $\xi_{uv} = \langle g, \varphi(\bar{u})\rangle - \langle g, \varphi(\bar{v})\rangle$ for $u, v \in e$ ($u \neq v$). Note that ξ_{uv} are Gaussian random variables with mean 0, and

$$\operatorname{Var}[\xi_{uv}] = \operatorname{Var}[\langle g, \varphi(\bar{u})\rangle - \langle g, \varphi(\bar{v})\rangle] = \|\varphi(\bar{u}) - \varphi(\bar{v})\|^2 \leq \operatorname{diam}(e)^2 .$$

The expectation of the maximum of (not necessarily independent) N Gaussian random variables with standard deviation bounded by σ is $O(\sqrt{\log N}\sigma)$. We have

$$\mathbb{E}\left[\tau_M - \tau_m\right] = \mathbb{E}\left[\max_{u,v\in e}(\xi_{uv})\right] = O\big(\sqrt{\log(|e|(|e|-1))}\operatorname{diam}(e)\big) = O\big(\sqrt{\log|e|}\operatorname{diam}(e)\big)$$

since the total number of random variables ξ_{uv} is $|e|(|e|-1)$. Therefore,

$$\Pr\left(\omega(u) \neq \omega(v) \text{ for some } u, v \in e\right) \leq \beta^{-1/2}\,\mathbb{E}\left[\tau_M - \tau_m\right] = O\big(\beta^{-1/2}\sqrt{\log|e|}\max_{u,v\in e}\|\varphi(\bar{u}) - \varphi(\bar{v})\|\big) .$$

We proved that ω satisfies the first property. Now we verify that ω satisfies the second property. Consider two vertices u and v with $\|\varphi(\bar{u}) - \varphi(\bar{v})\|^2 \geq \beta$. Given g, the random variable

$$Z = N(\langle g, \varphi(\bar{u})\rangle) - N(\langle g, \varphi(\bar{v})\rangle)$$

has Poisson distribution with rate $\lambda = |\langle g, \varphi(\bar{u})\rangle - \langle g, \varphi(\bar{v})\rangle|/\sqrt{\beta}$. We have

$$\Pr\left(Z \text{ is even} \mid g\right) = \sum_{k=0}^{\infty}\Pr\left(Z = 2k \mid g\right) = \sum_{k=0}^{\infty}\frac{e^{-\lambda}\lambda^{2k}}{(2k)!} = \frac{1+e^{-2\lambda}}{2} .$$

Note that λ is the absolute value of a Gaussian random variable with mean 0 and standard deviation $\sigma = \|\varphi(\bar{u}) - \varphi(\bar{v})\|/\sqrt{\beta} \geq 1$. Thus $\Pr(Z \text{ is even}) = \mathbb{E}\left[1 + e^{-2\sigma|\gamma|}\right]/2$, where γ is a standard Gaussian random variable, $\gamma \sim \mathcal{N}(0,1)$. We have

$$\Pr\left(\omega(u) \neq \omega(v)\right) = \mathbb{E}\left[\frac{1-e^{-2\sigma|\gamma|}}{2}\right] \geq \mathbb{E}\left[\frac{1-e^{-2|\gamma|}}{2}\right] \geq 0.3 . \qquad \square$$

Now we use the algorithm from Theorem 2.2 to obtain ℓ_2–ℓ_2^2 hypergraph orthogonal separators. The only difference is that we use the procedure from Lemma 5.3 rather than from Lemma 4.2 to generate assignments ω.

Theorem 5.4. *Random set S is an ℓ_2–ℓ_2^2 hypergraph m-orthogonal separator with distortions*

$$D_{\ell_2^2} = O(m) \qquad \text{and} \qquad D_{\ell_2}(r) = O\big(\beta^{-1/2}\sqrt{\log r}\, m\log m\log\log m\big),$$

probability scale $\alpha \geq 1/n$ and separation threshold $\beta \in (0,1)$.

Proof. The proof of the theorem is almost identical to that of Theorem 4.4. We first check conditions 1 and 2 of ℓ_2–ℓ_2^2 hypergraph orthogonal separators in the same way as we checked conditions 1 and 2 of hypergraph orthogonal separators in Theorem 4.4. When we verify that property 3 holds, we use bounds from Lemma 5.3. The only difference is how we upper bound the probability of the event $\mathcal{E}_2$. Recall that $\mathcal{E}_2$ is the event that

$$A = e\,, \quad A\cap S \neq \varnothing\,, \quad \text{and} \quad A \cap S \neq A\,.$$

We let $\rho_m = \min_{w\in e}\|\bar{w}\|^2$ as in Theorem 4.4. If $\mathcal{E}_2$ happens then

1. $r \le \rho_m$ since $A = e$, and
2. $W(u) \ne W(v)$ for some $u, v \in e$.

The probability that $r \le \rho_m$ is ρ_m. We upper bound the probability that $W(u) \ne W(v)$ for some $u, v \in e$. For each $i \in \{1, \dots, l\}$,

$$\begin{aligned}\Pr(\tilde{\omega}_i(u) \ne \tilde{\omega}_i(v) \text{ for some } u, v \in e) &\le O\big(\beta^{-1/2}\sqrt{\log|e|}\,\log\log m\big)\max_{u,v\in e}\|\varphi(\bar{u}) - \varphi(\bar{v})\| \\ &\le O\big(\beta^{-1/2}\sqrt{\log|e|}\,\log\log m\big)\max_{u,v\in e}\frac{\|\bar{u}-\bar{v}\|}{\min(\|\bar{u}\|,\|\bar{v}\|)} \\ &\le O\big(\beta^{-1/2}\sqrt{\log|e|}\,\log\log m\big)\times\rho_m^{-1/2}\times\max_{u,v\in e}\|\bar{u}-\bar{v}\|.\end{aligned}$$

By the union bound over $i \in \{1, \dots, l\}$, the probability that $W(u) \ne W(v)$ for some $u, v \in e$ is at most

$$O\big(l\times\beta^{-1/2}\sqrt{\log|e|}\,\log\log m\big)\times\rho_m^{-1/2}\times\max_{u,v\in e}\|\bar{u}-\bar{v}\|\,.$$

Therefore,

$$\begin{aligned}\Pr(\mathcal{E}_2) &\le \rho_m\times O\big(l\times\beta^{-1/2}\sqrt{\log|e|}\,\log\log m\big)\times\rho_m^{-1/2}\times\max_{u,v\in e}\|\bar{u}-\bar{v}\| \\ &\le O\big(l\times\beta^{-1/2}\sqrt{\log|e|}\,\log\log m\big)\times\rho_m^{1/2}\times\max_{u,v\in e}\|\bar{u}-\bar{v}\|\,.\end{aligned}$$

We get that the probability that e is cut by S is at most

$$\begin{aligned}\Pr(\mathcal{E}_1)+\Pr(\mathcal{E}_2) &\le \max_{u,v\in e}\|\bar{u}-\bar{v}\|^2 + O\big(l\times\beta^{-1/2}\sqrt{\log|e|}\,\log\log m\big)\,\rho_m^{1/2}\max_{u,v\in e}\|\bar{u}-\bar{v}\| \\ &\le \max_{u,v\in e}\|\bar{u}-\bar{v}\|^2 + O\big(l\times\beta^{-1/2}\sqrt{\log|e|}\,\log\log m\big)\min_{w\in e}\|\bar{w}\|\max_{u,v\in e}\|\bar{u}-\bar{v}\|.\end{aligned}$$

For $D_{\ell_2^2} = 1/\alpha$ and $D_{\ell_2}(r) = O\big(\beta^{-1/2}\sqrt{\log r}\,\log m\log\log m\big)/\alpha$, we get

$$\Pr(e \text{ is cut by } S) \le \alpha D_{\ell_2^2}\cdot\max_{u,v\in e}\|\bar{u}-\bar{v}\|^2 + \alpha D_{\ell_2}(|e|)\cdot\min_{w\in e}\|\bar{w}\|\cdot\max_{u,v\in e}\|\bar{u}-\bar{v}\|\,.$$

Note that $\alpha \ge 1/2^l \ge \Omega(1/m)$. Thus

$$D_{\ell_2^2} = O(m) \qquad \text{and} \qquad D_{\ell_2}(r) = O\big(\beta^{-1/2}\sqrt{\log r}\,m\log m\log\log m\big)\,. \qquad \square$$

6 Algorithm for Hypergraph Small-Set Expansion via ℓ_2–ℓ_2^2 hypergraph orthogonal separators

In this section, we present another algorithm for Hypergraph Small-Set Expansion. The algorithm finds a set with expansion proportional to $\sqrt{\phi^*_{G,\delta}}$. The proportionality constant depends on degrees of vertices and hyperedge size but not on the graph size. Here, we present our result for arbitrary hypergraphs. The result for uniform hypergraphs (Theorem 1.4) stated in the Introduction follows from our general result. In order to state our result for arbitrary graphs, we need the following definition.

Definition 6.1. Consider a hypergraph $H = (V,E)$. Suppose that for every edge e we are given a non-empty subset $e^\circ \subseteq e$. Let

$$\eta(u) = \sum_{e:u\in e^\circ} \frac{\log_2 |e|}{|e^\circ|} \qquad \text{and} \qquad \eta_{\max} = \max_{u\in V} \eta(u)\,.$$

Finally, let $\eta^H_{\max}$ be the minimum of $\eta_{\max}$ over all possible choices of subsets e°.

Claim 6.2.

1. $\eta^H_{\max} \leq \max_{u\in V} \sum_{e:u\in e} (\log_2 |e|)/|e|$.
2. *If H is an r-uniform graph with maximum degree $d_{\max}$ then $\eta^H_{\max} \leq (d_{\max} \log_2 r)/r$.*
3. *Suppose that we can choose one vertex in every edge so that no vertex is chosen more than once. Then $\eta^H_{\max} \leq \log_2 r_{\max}$, where $r_{\max}$ is the size of the largest hyperedge in H.*

Proof.

1. Let $e^\circ = e$ for every $e \in E$. We have $\eta^H_{\max} \leq \max_{u\in V} \sum_{e:u\in e} (\log_2 |e|)/|e|$.
2. By item 1,
$$\eta^H_{\max} \leq \max_{u\in V} \sum_{e:u\in e} (\log_2 |e|)/|e| = \max_{u\in V} \sum_{e:u\in e} (\log_2 r)/r = (d_{\max} \log_2 r)/r\,.$$
3. For every edge $e \in E$, let e° be the set that contains the vertex chosen for e. Then $|e^\circ| = 1$ and $|\{e : u \in e^\circ\}| \leq 1$ for every u. We have
$$\eta^H_{\max} \leq \max_{u\in V} \sum_{e:u\in e^\circ} \frac{\log_2 |e|}{|e^\circ|} \leq \max_{u\in V} \sum_{e:u\in e^\circ} \frac{\log_2 r_{\max}}{1} = \log_2 r_{\max}\,. \qquad \square$$

Theorem 6.3. *There is a randomized polynomial-time algorithm for the Hypergraph Small-Set Expansion problem that given a hypergraph $H = (V,E)$, and parameters $\varepsilon \in (0,1)$ and $\delta \in (0,1/2]$, finds a set $S \subset V$ of size at most $(1+\varepsilon)\delta n$ such that*

$$\begin{aligned}\phi(S) &\leq O_\varepsilon\left(\delta^{-1} \log \delta^{-1} \log\log \delta^{-1} \sqrt{\eta^H_{\max} \cdot \phi^*_{H,\delta}} + \delta^{-1} \phi^*_{H,\delta}\right) \\ &= \widetilde{O}_\varepsilon\left(\delta^{-1}\left(\sqrt{\eta^H_{\max} \phi^*_{H,\delta}} + \phi^*_{H,\delta}\right)\right).\end{aligned}$$

In particular, if H is an r-uniform hypergraph with maximum degree $d_{\max}$, then we have

$$\phi(S) \leq \widetilde{O}_\varepsilon\left(\delta^{-1}\left(\sqrt{d_{\max} \frac{\log_2 r}{r} \phi^*_{H,\delta}} + \phi^*_{H,\delta}\right)\right).$$

Proof. The proof is similar to that of Theorem 3.1. We solve the SDP relaxation for H-SSE and obtain an SDP solution $\{\bar{u}\}$. Denote the SDP value by $\mathsf{sdpcost}$. Consider an ℓ_2–ℓ_2^2 hypergraph orthogonal separator S with $m = 4/(\varepsilon\delta)$ and $\beta = \varepsilon/4$. Define a set S':

$$S' = \begin{cases} S, & \text{if } |S| \le (1+\varepsilon)\delta n, \\ \varnothing, & \text{otherwise.} \end{cases}$$

Clearly, $|S'| \le (1+\varepsilon)\delta n$. As in the proof of Theorem 3.1,

$$\Pr(u \in S') \in \left[(\alpha/2)\,\|\bar{u}\|^2, \alpha\|\bar{u}\|^2\right].$$

Note that

$$\Pr\big(S' \text{ cuts edge } e\big) \le \Pr\left(S \text{ cuts edge } e\right) \le \alpha D_{\ell_2} \max_{u,v\in e} \|\bar{u}-\bar{v}\|^2 + \alpha D_{\ell_2}(r) \min_{w\in e} \|\bar{w}\| \max_{u,v\in e} \|\bar{u}-\bar{v}\|.$$

Let $\mathcal{C} = \alpha^{-1}\,\mathbb{E}\left[|E_{\mathrm{cut}}(S')|\right]$. Let

$$Z = |S'| - \frac{|E_{\mathrm{cut}}(S')|}{4\mathcal{C}}.$$

We have

$$\mathbb{E}[Z] = \mathbb{E}\left[|S'|\right] - \mathbb{E}\left[\frac{|E_{\mathrm{cut}}(S')|}{4\mathcal{C}}\right] \ge \sum_{u\in V} \frac{\alpha}{2}\cdot\|\bar{u}\|^2 - \frac{\alpha}{4} = \frac{\alpha}{2} - \frac{\alpha}{4} = \frac{\alpha}{4}.$$

Now we upper bound $\mathcal{C}$. Consider the optimal choice of e° for H in the definition of $\eta^H_{\max}$.

$$\begin{aligned}
\mathcal{C} &= \alpha^{-1}\,\mathbb{E}\left[|E_{\mathrm{cut}}(S')|\right] \le \alpha^{-1}\sum_{e\in E} \Pr\left(e \text{ is cut by } S\right) \\
&\le D_{\ell_2^2} \sum_{e\in E} \max_{u,v\in e} \|\bar{u}-\bar{v}\|^2 + \sum_{e\in E} D_{\ell_2}(|e|) \min_{w\in e} \|\bar{w}\| \max_{u,v\in e} \|\bar{u}-\bar{v}\| \\
&\le D_{\ell_2^2}\cdot \mathsf{sdpcost} + \sum_{e\in E} D_{\ell_2}(|e|) \sum_{w\in e^\circ} \left(\frac{\|\bar{w}\|}{|e^\circ|}\right) \times \max_{u,v\in e} \|\bar{u}-\bar{v}\| \\
&\le D_{\ell_2^2}\cdot \mathsf{sdpcost} + \sum_{e\in E}\sum_{w\in e^\circ} \frac{D_{\ell_2}(|e|)\|\bar{w}\|}{\sqrt{|e^\circ|}} \times \frac{\max_{u,v\in e}\|\bar{u}-\bar{v}\|}{\sqrt{|e^\circ|}} \\
&\le D_{\ell_2^2}\cdot \mathsf{sdpcost} + \sqrt{\sum_{e\in E}\sum_{w\in e^\circ} \frac{D_{\ell_2}(|e|)^2\|\bar{w}\|^2}{|e^\circ|}}\sqrt{\sum_{e\in E}\sum_{w\in e^\circ} \frac{\max_{u,v\in e}\|\bar{u}-\bar{v}\|^2}{|e^\circ|}} \qquad (6.1) \\
&\le D_{\ell_2^2}\cdot \mathsf{sdpcost} + \sqrt{\sum_{w\in V}\left(\|\bar{w}\|^2 \sum_{e:w\in e^\circ} \frac{D_{\ell_2}(|e|)^2}{|e^\circ|}\right)}\sqrt{\mathsf{sdpcost}},
\end{aligned}$$

where the inequality of line (6.1) follows from Cauchy–Schwarz. For every vertex w,

$$\sum_{e:w\in e^\circ} \frac{D_{\ell_2}(|e|)^2}{|e^\circ|} \le O_\beta(m\log m\log\log m)^2 \sum_{e:w\in e^\circ} \frac{\log_2|e|}{|e^\circ|} \le O_\beta(m\log m\log\log m)^2 \times \eta^H_{\max}$$

and

$$\sum_{w\in V} \|\bar{w}\|^2 = 1.$$

Therefore,

$$\mathcal{C} \le O_\beta\left(m\mathsf{sdpcost} + m\log m \log\log m\sqrt{\eta^H_{\max}\cdot \mathsf{sdpcost}}\right).$$

By the argument from Theorem 3.1, we get that if we sample S' sufficiently many times (i. e., $(4n^2/\alpha)$ times), we will find a set S' such that

$$\frac{|E_{\text{cut}}(S')|}{|S'|} \le 4\mathcal{C} \le O_\beta\left(\delta^{-1}\log\delta^{-1}\log\log\delta^{-1}\sqrt{\eta^H_{\max}\cdot \mathsf{sdpcost}} + \delta^{-1}\mathsf{sdpcost}\right)$$

with probability exponentially close to 1. □

7 SDP integrality gap

In this section, we present an integrality gap for the SDP relaxation for H-SSE. We also give a lower bound on the distortion of a hypergraph m-orthogonal separator.

Theorem 7.1. *For $\delta = 1/r$, the integrality gap of the SDP for H-SSE is at least $1/(2\delta) = r/2$.*

Proof. Consider a hypergraph $H = (V,E)$ on $n = r$ vertices with one hyperedge $e = V$ (e contains all vertices). Note that the expansion of every set of size $\delta n = 1$ is 1. Thus $\phi^*_{H,\delta} = 1$.

Consider an SDP solution that assigns vertices mutually orthogonal vectors of length $1/\sqrt{r}$. It is easy to see this is a feasible SDP solution. Its value is $\max_{u,v\in e}\|\bar{u}-\bar{v}\|^2 = 2/r$. Therefore, the SDP integrality gap is at least $r/2$. □

Now we give a lower bound on the distortion of hypergraph m-orthogonal separators.

Lemma 7.2. *For every $m > 4$, there is an SDP solution such that every hypergraph m-orthogonal separator with separation threshold $\beta < 2$ has distortion at least $\lceil m\rceil/4$.*

Proof. Consider the SDP solution from Theorem 7.1 for $n = r = \lceil m\rceil$. Consider a hypergraph m-orthogonal separator S for this solution. Let D be its distortion. Note that condition (2) from the definition of hypergraph orthogonal separators applies to any pair of distinct vertices (u,v) since $\|\bar{u}-\bar{v}\|^2 = 2\|u\|^2 = 2\|v\|^2$.

By the inclusion–exclusion principle, we have

$$\begin{aligned}\Pr(|S| = 1) &\ge \sum_{u\in S}\Pr(u\in S) - \frac{1}{2}\sum_{u,v\in S, u\ne v}\Pr(u\in S, v\in S)\\ &\ge \sum_{u\in S}\alpha\|\bar{u}\|^2 - \frac{1}{2}\sum_{u,v\in S,u\ne v}\frac{\alpha\min(\|\bar{u}\|^2,\|\bar{v}\|^2)}{m} = \alpha - \frac{\alpha n(n-1)}{2mr} \ge \alpha/2\,.\end{aligned}$$

On the other hand, if $|S| = 1$ then S cuts e. We have

$$\Pr(|S| = 1) \le \Pr(S \text{ cuts } e) \le \alpha D\max_{u,v\in e}\|\bar{u}-\bar{v}\|^2 = 2\alpha D/r\,.$$

We get that $\alpha/2 \le 2\alpha D/r$ and thus $D \ge r/4 = \lceil m\rceil/4$. □

8 Reduction from Vertex Expansion to Hypergraph Expansion

In the reduction from vertex expansion to hypergraph expansion, we will use the notion of *Symmetric Vertex Expansion*. For a graph $G = (V,E)$ and set $S \subset V$, we define its outer neighborhood $N(S)$ as follows.

$$N(S) = \{u \in \bar{S} : \exists v \in S \text{ such that } \{u,v\} \in E\}.$$

The symmetric vertex expansion of a set, denoted by $\Phi^V(S)$, is defined as

$$\Phi^V(S) = \frac{|N(\bar{S}) \cup N(S)|}{\min(|S|,|\bar{S}|)} \qquad \text{and} \qquad \Phi^V_{G,\delta} = \min_{\substack{S \subset V \\ 0<|S|\le \delta n}} \Phi^V(S).$$

We will use the following reduction from vertex expansion to symmetric vertex expansion.

Theorem 8.1 (Louis, Raghavendra, and Vempala [20])**.** *Given a graph G, there exists a graph G' such that*

$$c_1 \phi^V_{G,\delta} \le \Phi^V_{G',\delta} \le c_2 \phi^V_{G,\delta}$$

where $c_1, c_2 > 0$ are absolute constants, and the maximum degree of graph G' is equal to the maximum degree of graph G. Moreover, there exists a polynomial time algorithm to compute such graph G'.

The following result is a detailed version of the informal Theorem 1.6 stated in the Introduction.

Theorem 8.2 ("Graph to Hypergraph Theorem," detailed statement)**.** *There exist absolute constants $c_1, c_2 \in \mathbb{R}^+$ such that for every graph $G = (V,E)$, there exists a polynomial-time computable hypergraph $H = (V',E')$ such that $c_1\phi^*_{H,\delta} \le \phi^V_{G,\delta} \le c_2\phi^*_{H,\delta}$. Also, for the hypergraph H, we have $\eta^H_{\max} \le \log_2(d_{\max}+1)$, where $\eta^H_{\max}$ is defined in Definition 6.1 and $d_{\max}$ is the maximum degree of G.*

Proof. Starting with graph G, we use Theorem 8.1 to obtain a graph $G' = (V',E')$ such that

$$c_1 \phi^V_{G,\delta} \le \Phi^V_{G',\delta} \le c_2 \phi^V_{G,\delta}.$$

Next we construct hypergraph $H = (V',E'')$: for every vertex $v \in V'$, we add the hyperedge $\{v\} \cup N(\{v\})$ to E'' (note that $N(\{v\})$ is the set of neighbors of v in G).

Fix an arbitrary set $S \subset V$. We first show that $\Phi^V(S) \le \phi_H(S)$. Consider the set of vertices $N(\bar{S})$. Each vertex in $v \in N(\bar{S})$ has a neighbor, say u, in $\bar{S}$. Therefore the hyperedge $\{v\} \cup N(\{v\})$ is cut by S in H. Similarly, for each vertex $v \in N(S)$, the hyperedge $\{v\} \cup N(\{v\})$ is cut by S in H. Therefore,

$$\Phi^V(S) \le \frac{|N(\bar{S})| + |N(S)|}{|S|} = \frac{|E_{\text{cut}}(S)|}{|S|} \le \phi_H(S).$$

Now we verify that $\phi_H(S) \le \Phi^V(S)$. For any hyperedge $(\{v\} \cup N(\{v\})) \in E_{\text{cut}}(S)$, the vertex v has to belong to either $N(\bar{S})$ or $N(S)$. Therefore,

$$\phi_H(S) \le \frac{|E_{\text{cut}}(S)|}{|S|} \le \frac{|N(\bar{S})| + |N(S)|}{|S|} = \Phi^V(S).$$

Thus we get that $\phi_H(S) = \Phi^V(S)$ for every $S \subset V$, and hence $\phi^*_{H,\delta} = \Phi^V_{G',\delta}$. Therefore, by Theorem 8.1,

$$c_1 \phi^V_{G,\delta} \le \phi^*_{H,\delta} \le c_2 \phi^V_{G,\delta}.$$

Finally, we upper bound $\eta^H_{\max}$. We use part 3 of Claim 6.2. We choose vertex v in the hyperedge $\{v\} \cup N(\{v\})$. By Claim 6.2, $\eta^H_{\max} \le \log_2 r_{\max}$, where $r_{\max}$ is the size of the largest hyperedge. Note that $|\{v\} \cup N(\{v\})| = \deg v + 1$. Thus

$$\eta^H_{\max} \le \log_2 r_{\max} \le \log_2(d_{\max} + 1)\,. \qquad \square$$

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AUTHORS

Anand Louis
Assistant Professor
Department of Computer Science and Automation
Indian Institute of Science
Bengaluru
anand@csa.iisc.ernet.in
http://drona.csa.iisc.ernet.in/~anand/

Yury Makarychev
Associate Professor
Toyota Technological Institute at Chicago
Chicago, IL
yury@ttic.edu
http://ttic.uchicago.edu/~yury/

ABOUT THE AUTHORS

ANAND LOUIS is an Assistant Professor in the Department of Computer Science and Automation at the Indian Institute of Science, Bengaluru. He received his Ph. D. from Georgia Tech under the supervision of Santosh Vempala in 2014. Subsequently he was a postdoctoral research associate in the Department of Computer Science at Princeton University for two years.

YURY MAKARYCHEV is an Associate Professor at the Toyota Technological Institute at Chicago. He received his Ph. D. from Princeton University under the supervision of Moses Charikar in 2008. Yury did a postdoctoral fellowship at Microsoft Research and then joined TTIC in 2009.