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NOTE

On the Hardness of Approximating the k-WAY HYPERGRAPH CUT Problem

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Abstract. We consider the approximability of the k-WAY HYPERGRAPH CUT problem: the input is an edge-weighted hypergraph $G = (V, \mathcal{E})$ and an integer k and the goal is to remove a min-weight subset of the edges such that the residual hypergraph has at least k connected components. When G is a graph this problem admits a $2(1-1/k)$-approximation (Saran and Vazirani, SIAM J. Comput. 1995). However, there has been no non-trivial approximation ratio for general hypergraphs. In this note we show, via a very simple reduction, that an α-approximation for k-WAY HYPERGRAPH CUT implies an α^2-approximation for the DENSEST ℓ-SUBGRAPH problem. Our reduction combined with the hardness result of (Manurangsi, STOC'17) implies that under the Exponential Time Hypothesis (ETH), there is no $n^{1/(\log\log n)^c}$-approximation for k-WAY HYPERGRAPH CUT where $c > 0$ is a universal constant and n is the number of nodes.

k-WAY HYPERGRAPH CUT is a special case of k-WAY SUBMODULAR PARTITION and hence our hardness applies to this latter problem as well. These hardness results are in contrast to a 2-approximation for closely related problems where the goal is to separate k given terminals (Chekuri and Ene, FOCS'11), (Ene et al., SODA'13).

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1 Introduction

We consider the following problem.

k-WAY HYPERGRAPH CUT: Let $G = (V, \mathcal{E})$ be a hypergraph with edge weights given by $w : \mathcal{E} \to \mathbb{R}_+$. Given an integer k, find a min-weight subset of edges $\mathcal{E}' \subseteq \mathcal{E}$ such that $G - \mathcal{E}'$ has at least k connected components. Equivalently, find a partition of V into k non-empty sets $V_1, V_2, \ldots, V_k$ such that the weight of the hyperedges that cross the partition[1] is minimized.

k-WAY HYPERGRAPH CUT is known as the k-CUT problem when the input is a graph and is one of the well-studied graph partitioning problems. k-WAY HYPERGRAPH CUT is a special case of a more general *submodular* partitioning problem defined below.[2]

k-WAY SUBMODULAR PARTITION (k-WAY SUB-MP): Let $f : 2^V \to \mathbb{R}_+$ be a non-negative submodular set function[3] over a finite ground set V. The k-way submodular partition problem is to find a partition $V_1, \ldots, V_k$ of V to minimize $\sum_{i=1}^{k} f(V_i)$ under the condition that each part $V_i \neq \emptyset$. An important special case is when f is symmetric and we refer to it as k-WAY SYM-SUB-MP.

Throughout the paper we will assume that the parameter k is part of the input and can be a function of the other input parameters such as the size of the hypergraph. Several of the problems are also interesting when k is fixed to some absolute constant. In this case we will explicitly use the terminology "fixed k" to distinguish it from the general case. The result in this paper is on the hardness of approximating k-WAY HYPERGRAPH CUT when k is part of the input; we note that if k is a fixed constant there is a polynomial-time algorithm for k-WAY HYPERGRAPH CUT due to recent work [4, 3]. We seek to establish a hardness factor for k-WAY HYPERGRAPH CUT that is polynomial in the number of nodes of the underlying hypergraph; this is only feasible for instances in which k is also polynomial in the hypergraph size. This will be implicit in the reduction.

We refer the reader to [14, 5] to see why k-WAY HYPERGRAPH CUT is a special case of k-WAY SUB-MP. The k-CUT problem is not only a special case of k-WAY HYPERGRAPH CUT but it is also a special case of k-WAY SYM-SUB-MP. When k is part of the input, k-CUT is NP-Hard [8] and hence all the problems we discussed so far are also NP-Hard. k-WAY SYM-SUB-MP admits a $2(1-1/k)$-approximation [15, 19] and hence also k-CUT [17]. For k-WAY HYPERGRAPH CUT a $2\Delta(1-1/k)$-approximation easily follows from the $2(1-1/k)$-approximation for k-CUT; here Δ is the rank of the hypergraph (the maximum size of any hyperedge). On the other hand, in the general case, the known approximation algorithms for k-WAY HYPERGRAPH CUT and k-WAY SUB-MP provide an approximation ratio of $(k-1)$ [19]. Despite a claim of APX-Hardness for k-CUT in [17] (attributed to Papadimitriou), no proof has been published in the literature; Manurangsi [13] showed that k-CUT does not admit a $(2-\varepsilon)$-factor approximation for any fixed $\varepsilon > 0$ under the Small Set Expansion Hypothesis. As far as we are aware, prior to our work, no better hardness result was known for k-WAY HYPERGRAPH CUT.

[1] A hyperedge e crosses a partition of the vertex set if e properly intersects at least two parts of the partition.

[2] The names of several problems in this paper such as k-WAY HYPERGRAPH CUT and k-CUT include k as a prefix while k also is an input parameter. Although there is some scope for confusion due to this notational overload, we stick to it for historical reasons and believe that it is not hard to disambiguate based on the context.

[3] A set function $f : 2^V \to \mathbb{R}$ is *submodular* if $f(A) + f(B) \geq f(A \cap B) + f(A \cup B)$ for all $A, B \subseteq V$. Moreover, f is *symmetric* if $f(A) = f(V - A)$ for all $A \subseteq V$.

In this note we show that a good approximation for k-WAY HYPERGRAPH CUT would imply a good approximation for the DENSEST ℓ-SUBGRAPH problem[4] which has been extensively investigated and has been shown to be conditionally hard.

DENSEST ℓ-SUBGRAPH: Given a graph $G=(V,E)$ and an integer ℓ, find a subset $S \subseteq V$ of ℓ nodes to maximize the number of edges in the induced subgraph $G[S]$.

The current best polynomial-time approximation ratio for DENSEST ℓ-SUBGRAPH is $O(n^{1/4+\varepsilon})$ [1]; note that an ℓ-approximation is easy. Although the problem is expected to be quite hard to approximate, the known hardness results are weak; a PTAS for DENSEST ℓ-SUBGRAPH can be ruled out only under the assumption that NP $\not\subseteq \bigcap_{\varepsilon>0}$ BPTIME$(2^{n^{\varepsilon}})$ [11]. Polynomial-factor integrality gaps for several strong SDP relaxations are known [2]. In a breakthrough result, Manurangsi [12] showed that under the Exponential Time Hypothesis (ETH), DENSEST ℓ-SUBGRAPH is hard to approximate to a factor better than $n^{1/(\log\log n)^c}$ where n is the number of nodes in the input graph and $c>0$ is a universal constant. We state his result more precisely in the theorem below.

Theorem 1.1 (Manurangsi)**.** *There exists a constant $c>0$ such that the following holds, assuming ETH. No polynomial-time algorithm can, given a graph G with n vertices and a positive integer $\ell \leq n$, distinguish between the following two cases:*

- *G contains an ℓ-clique as a subgraph.*
- *Every ℓ-node subgraph of G has at most $\binom{\ell}{2}/n^{1/(\log\log n)^c}$ edges.*

To formally state our result it is more convenient to relate the approximation ratio to the parameter s of a given graph (or hypergraph) which is the sum of the number of nodes and edges (or hyperedges). The above theorem gives an $s^{1/(\log\log s)^{c'}}$-hardness for DENSEST ℓ-SUBGRAPH for some $c'>0$, since s and n are polynomially related for graphs. On the other hand, the tight instances for the algorithm of [1] for DENSEST ℓ-SUBGRAPH have $|E|=\Theta(|V|^{3/2})$. For these instances, it is not known how to obtain an approximation ratio better than $O(|V|^{1/4})=O(s^{1/6})$. Our main result is the following.

Theorem 1.2. *Let $\alpha:\mathbb{N}\to\mathbb{R}_+$ be a function. A polynomial-time $\alpha(s)$-approximation algorithm for k-WAY HYPERGRAPH CUT, where s is the size of the input hypergraph, implies a polynomial-time $2(\alpha(t+1))^2$-approximation algorithm for DENSEST ℓ-SUBGRAPH where t is size of the input graph.*

Corollary 1.3. *Assuming ETH, there is no $s^{1/(\log\log s)^c}$-approximation for k-WAY HYPERGRAPH CUT, where $c>0$ is a universal constant and s is size of the input hypergraph. In particular, assuming ETH, there is no $n^{1/(\log\log n)^c}$-approximation for k-WAY HYPERGRAPH CUT, where $c>0$ is some universal constant and n is the number of nodes in the input hypergraph.*

Proof. The first part follows by combining the preceding theorem with Theorem 1.1. For the second part we observe that the reduction in the proof of Theorem 1.2 creates instances of k-WAY HYPERGRAPH CUT in which s is not greater than a fixed polynomial in n where n is the number of nodes of the hypergraph. The second part follows from this. □

[4] DENSEST ℓ-SUBGRAPH problem is typically referred to as the DENSEST k-SUBGRAPH problem. We use ℓ since k is already used for k-WAY HYPERGRAPH CUT and several related problems.

When k is a fixed constant, one can reduce k-WAY HYPERGRAPH CUT and k-WAY SUB-MP to solving $O(n^{k-1})$ instances of the "terminal" version of these problems which have a $2(1-1/k)$ approximation. We refer the readers to [5, 7] for more details on these related problems.

2 Proof of Theorem 1.2

Let $(G=(V,E),\ell)$ be an instance of DENSEST ℓ-SUBGRAPH. We construct a hypergraph $H=(A,\mathcal{F})$ as follows. For each edge $e\in E$ we create a node a_e and add it to A. Moreover we add a new special node r to A. Thus $A=\{r\}\cup\{a_e \mid e\in E\}$. For each node $v\in V$ we add a hyperedge f_v to $\mathcal{F}$ where $f_v=\{r\}\cup\{a_e \mid e\in\delta_G(v)\}$ where $\delta_G(v)$ is the set of edges in E that are incident to v in G. Thus H is basically the hypergraph obtained from G by flipping the role of nodes and edges and then adding the extra node r to each hyperedge. We also observe that $|A|+|\mathcal{F}|=1+|V|+|E|=s+1$.

For a subset $S\subseteq V$, we let $E_G(S)$ denote the set of edges in E with both endpoints in S. The following is a simple but useful claim about the relationship between G and H.

Claim 2.1. *For any $1\le q\le |V|$, if there is a set $S\subseteq V$ with $|S|=q$ and $|E_G(S)|=Q-1$ then the k-WAY HYPERGRAPH CUT instance on H with $k=Q$ has a cut of value at most q. Moreover, given any $F\subseteq\mathcal{F}$ of size $|F|=q'$ such that $H-F$ has Q' connected components, there is a subset $S'\subseteq V$ such that $|S'|=q'$ and $|E_G(S')|=Q'-1$.*

Proof. Consider a set $F\subseteq\mathcal{F}$ of hyperedges in H. Suppose we remove them from H. Let $V_F=\{v\in V \mid f_v\in F\}$ be the nodes in G that correspond to the hyperedges in F. Then a node $a_e\in A$ corresponding to an edge $e=uv$ is separated from r in H iff both $u,v\in V_F$; in this case the node a_e becomes an isolated node in $H-F$. Thus the number of connected components in $H-F$ is precisely equal to $|E_G(V_F)|+1$. This correspondence proves both parts of the claim. □

Suppose we have an $\alpha(s)$-approximation for k-WAY HYPERGRAPH CUT. We obtain an approximation for DENSEST ℓ-SUBGRAPH as follows. Let (G,ℓ) be a given instance of DENSEST ℓ-SUBGRAPH. First assume that we know the optimum solution value L for the given instance. We construct the hypergraph H as described earlier, and give H and $k=L+1$ to the $\alpha(s)$-approximation algorithm for k-WAY HYPERGRAPH CUT. By Claim 2.1 there is an optimum solution to the k-WAY HYPERGRAPH CUT instance on H of value at most ℓ.

Thus, the approximation algorithm will output a set $F\subseteq\mathcal{F}$ such that $|F|\le\alpha(s+1)\cdot\ell$ such that $H-F$ has at least $L+1$ connected components. For simplicity, we let $\alpha=\alpha(s+1)$. By the second part of the claim we can obtain a set $S'\subseteq V$ such that $|S'|\le\alpha\cdot\ell$ and $|E_G(S')|\ge L$. Then we shall output a random subset $S'\subseteq S$ of size ℓ as the solution for the DENSEST ℓ-SUBGRAPH problem. Then, the expected number of edges induced by S is

$$\begin{aligned}|E_G(S')|\cdot\frac{\ell}{|S'|}\cdot\frac{\ell-1}{|S'|-1}\ge L\cdot\frac{\ell}{\alpha\cdot\ell}\cdot\frac{\ell-1}{\alpha\cdot\ell-1}&=\frac{L}{\alpha}\left(\frac{1}{\alpha}-\frac{1-1/\alpha}{\alpha\ell-1}\right)\\&\ge\frac{L}{\alpha}\left(\frac{1}{\alpha}-\frac{1}{\alpha\ell-\alpha}\right)=\frac{\ell-2}{\ell-1}\cdot\frac{L}{\alpha^2}.\end{aligned}$$

In the above sequence, we used $\alpha \geq 1$ and assumed $\ell \geq 2$. One can indeed efficiently and deterministically find a set $S \subseteq V$ of size ℓ such that $|E_G(S)| \geq \frac{\ell-2}{\ell-1} \cdot \frac{L}{\alpha^2}$, using the method of conditional expectations. This holds since conditioned on the event that S contains a given set of vertices, the expectation of $|E_G(S)|$ can be computed easily. Since L is the optimum value for the given instance of DENSEST ℓ-SUBGRAPH, we obtain the desired $\frac{\ell-1}{\ell-2} \cdot \alpha^2 = \frac{\ell-1}{\ell-2} \cdot (\alpha(s+1))^2$-approximation. The assumption that the algorithm knows the value L can be easily removed by trying all possible values of L from 0 to $|E(G)|$. This completes the proof of Theorem 1.2.

3 Discussion and open problems

We proved conditional hardness of k-WAY HYPERGRAPH CUT. An important open question is to obtain hardness of approximation for k-WAY HYPERGRAPH CUT under the standard $P \neq NP$ assumption. At this point we do not even have APX-Hardness. For k-WAY SYM-SUB-MP Santiago [16] has shown an exponential lower bound on the number of value oracle queries required to obtain an approximation ratio strictly below 2. Can one show exponential query lower bounds for k-WAY SUB-MP even for super-constant approximation factors? This question was raised in [16] based on the result in this paper.

For any fixed constant k, k-CUT in graphs can be solved in polynomial time [8]; there are several different algorithms for this problem by now and we refer the reader to [6, 10] for a discussion of recent work and other pointers. It was an open problem whether k-WAY HYPERGRAPH CUT can be solved in polynomial time when k is a fixed constant. A randomized polynomial-time algorithm was developed in [4], and recently a deterministic algorithm was obtained in [3]. The complexity status of k-WAY SUB-MP is open for any fixed $k > 3$; for $k \leq 3$ there is a polynomial-time algorithm [14] building upon [18]. For k-WAY SYM-SUB-MP a polynomial-time algorithm is known for $k \leq 4$ [9] and the complexity status is open for any fixed $k > 4$.

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