THEORY OF COMPUTING, Volume 19 (6), 2023, pp. 1–21
www.theoryofcomputing.org

SPECIAL ISSUE: APPROX-RANDOM 2020

Reaching a Consensus on Random Networks: The Power of Few

Linh Tran Van Vu

Received March 1, 2021; Revised July 20, 2022; Published November 14, 2023

Abstract. A community of n individuals splits into two camps, Red and Blue. The individuals are connected by a social network, which influences their colors. Every day each person changes their color according to the majority of their neighbors. Red (Blue) wins if everyone in the community becomes Red (Blue) at some point.

We study this process when the underlying network is the random Erdős–Rényi graph $G(n, p)$. With a balanced initial state ($n/2$ persons in each camp), it is clear that each color wins with the same probability.

Our study reveals that for any constants p and ε, there is a constant c such that if one camp has at least $n/2 + c$ individuals at the initial state, then it wins with probability at least $1 - \varepsilon$. The surprising fact here is that c *does not* depend on n, the population of the community. When $p = 1/2$ and $\varepsilon = .1$, one can set $c = 5$, meaning one camp has $n/2 + 5$ members initially. In other words, it takes only 5 extra people to win an election with overwhelming odds. We also generalize the result to $p = p_n = o(1)$ in a separate paper.

A preliminary version of this paper appeared in the Proceedings of the 24th International Conference on Randomization and Computation (RANDOM'20) [16].

ACM Classification: G.3, G.2.2

AMS Classification: 37E25, 05C57, 05C80, 82B43, 68R10, 60K35, 60K37

Key words and phrases: random graphs, Erdős-Rényi model, majority dynamics, consensus

DOI: 10.4086/toc.2023.v019a006

1 Introduction

1.1 The opinion exchange dynamics

Building mathematical models to explain how collective opinions are formed is an important and interesting task (see [13] for a survey on the topic, with examples from various fields, economy, sociology, statistical physics, to mention a few).

Obviously, our opinions are influenced by people around us, and this motivates the study of the following natural and simple model: A community of n individuals splits into two camps, Red and Blue, representing two competing opinions, which can be on any topic such as brand competition, politics, ethical issues, etc. The individuals are connected by a social network, which influences their opinion on a daily basis (by some specific rule). We say that Red (respectively Blue) *wins* if everyone in the community becomes Red (respectively Blue) at some point.

We study this process when the underlying network is random. In this paper, we focus on the Erdős–Rényi random graph $G(n,p)$, which is the most popular model of random graphs [4, 10]. We use the majority rule, which is a natural choice. When a new day comes, a vertex scans its neighbors' colors from the previous day and adopts the dominant one. If there is a tie, it keeps its color.

Definition 1.1. The random graph $G(n,p)$ on $n \in \mathbb{N}$ vertices and density $p \in (0,1)$ is obtained by putting an edge between any two vertices with probability p, independently.

1.2 Results

With a balanced initial state ($n/2$ persons in each camp), by symmetry, each color wins with the same probability $q < 1/2$, regardless of p. (Notice that there are graphs, such as the empty and complete graphs, on which no one wins.) Our study reveals that for any given p and ε, there is a constant c such that if one camp has at least $n/2 + c$ individuals at the initial state, then it wins with probability at least $1 - \varepsilon$. The surprising fact here is that c does not depend on n, the population of the community. When $p = 1/2$ and $\varepsilon = .1$, one can set c as small as 5.

Theorem 1.2 (The power of few)**.** *Consider the (majority) process on $G(n,1/2)$. Assume that the Red camp has at least $n/2+5$ vertices at the initial state, where $n \geq 1000$. Then Red wins after the fourth day with probability at least 90%.*

This result can be stated without the Erdős–Rényi $G(n,1/2)$ model; one can state an equivalent theorem by choosing the network uniformly, from the set of all graphs on n vertices.

This result reveals an interesting phenomenon, which we call "the power of few." The collective outcome can be extremely sensitive, as a modification of the smallest scale in the initial setting leads to the opposite outcome.

Our result applies in the following equivalent settings.

Model 1. We fix the two camps, of sizes $n/2 + c$ and $n/2 - c$, and draw a random graph $G \sim G(n,p)$ over their union.

Models 2. We draw a random graph $G \sim G(n, p)$ first, let Red be a random subset of $n/2 + c$ vertices (chosen uniformly from all subsets of that size), and Blue be the rest.

Model 3. We split the society into two camps of size $n/2$ each, then draw the random graph $G \sim G(n, p)$ on their union, then recolor c randomly selected Blue vertices to Red.

Model 4. Split the society into two camps (Red and Blue) of size $n/2 - c$ each and a "swing" group (with no color yet) of $2c$ individuals. Draw the random graph on their union. Now let the swing group join the Red camp.

With Model 3, we can imagine a balanced election process at the beginning. Then $c = 5$ people move to the other camp. Theorem 1.2 asserts that this tiny group already guarantees the final win with overwhelming odds. Similarly, Model 4 implies that a swing group of size 10 decides the outcome.

Our result can also be used to model the phenomenon that outcomes in seemingly identical situations become opposite. Consider two communities, each has exactly n individuals, sharing the same social network. In the first community, the Red camp has size $n/2 + c$, and the Blue camp has $n/2 - c$. In the second community, the Blue camp has $n/2 + c$ and the Red camp has $n/2 - c$. If n is large, there is no way to tell the difference between the two communities. Even if we record everyone's initial opinion, clerical errors will surely swallow the tiny difference of $2c$. However, at the end, the collective opinion will be opposite, with high probability.

Now we state the general result for arbitrary constant density p.

Theorem 1.3 (Asymptotic bound)**.** *Let p be a constant in $(0, 1)$ and c_n be a positive integer which may depend on n . Assume that Red has $n/2 + c_n$ in day 0 and the random graph is $G(n, p)$. Then Red wins after four days with probability at least $1 - K(p) \max\{n^{-1}, c_n^{-2}\}$, where $K(p)$ depends only on p.*

Both results follow from Theorem 1.6, which, in a slightly technical form, describes how the process evolves day by day. Our results can be extended to cover the case when there are more than 2 opinions; details will appear in a later paper [17].

1.3 Related results

Our problem is related to a well-studied class of *opinion exchange dynamics* problems. In the field of Computer Science, loosely-related processes are studied in *population protocols* [2, 1], where individuals/agents/nodes choose their next state based on that of their neighbors. The most separating difference is the network, as connections in these models often randomly change with time, while our study concerns a fixed network (randomly generated before the process begins).

The survey by Mossel and Tamuz [13] discussed several models for these problems, including the *DeGroot model* [6], where an individual's next state is a weighted average of its neighbors' current states, the *voter model* [5], where individuals change states by emulating a random neighbor each day. The *majority dynamics* model is in fact the same as ours, and is also more popular than the other two, having been studied in [12, 8, 3]. The key difference, as compared to our study, is in the setups. In these earlier papers, each individual chooses his/her initial color uniformly at random. The central limit theorem thus guarantees that with high probability, the initial difference between the two camps is of order $\Theta(\sqrt{n})$. Therefore, these papers did not touch upon the "power of few" phenomenon, which is our key message. On the other hand, they considered sparse random graphs where the density $p = p_n \to 0$ as $n \to +\infty$.

In [3], Benjamini, Chan, O'Donnell, Tamuz, and Tan considered random graphs with $p \geq \lambda n^{-1/2}$, where λ is a sufficiently large constant, and showed that the dominating color wins with probability at least .4 [3, Theorem 1.2], while conjecturing that this probability in fact tends to 1 as $n \to \infty$. This conjecture was proved by Fountoulakis, Kang, and Makai [8, Theorem 1.1].

Theorem 1.4 (Fountoulakis, Kang, Makai)**.** *For any $0 < \varepsilon \leq 1$ there is $\lambda = \lambda(\varepsilon)$ such that the following holds for $p \geq \lambda n^{-1/2}$: With probability at least $1 - \varepsilon$, over the choice of the random graph $G(n, p)$ and the choice of the initial state, the dominating color wins after four days.*

For related results on random regular graphs, see [12, 13].

1.4 Extension for sparse random graphs

Note that the results presented in this paper only apply to a constant p, which, in the context of $G(n, p)$, produces *dense graphs*. For *sparse graphs*, i. e., when $p = p_n$ depends on n and tends to 0 as $n \to +\infty$, the main ideas in this paper can be used, but with different algebraic techniques, to obtain a similar result.

Theorem 1.5. *For any $0 < \varepsilon \leq 1$ and $\lambda > 0$ there is $c = c(\varepsilon, \lambda)$ such that for $p \geq (1 + \lambda)(\log n)/n$, if Red starts with $n/2 + c/p$ members, then it wins with probability at least $1 - \varepsilon$.*

The technical changes needed to prove this theorem require rewriting entire proofs with new computations, so we leave the proof to our future paper [17]. Additional information such as the length of the process and the explicit relation between the bound with p and c will also be discussed there. Notice that when p is a constant, this result covers the "Power of Few" phenomenon as a special case, albeit with c much larger than 5. Thus, the techniques and results in this paper still have merit since they achieve a specific, surprisingly small constant. Theorem 1.5 no longer holds for $p < (\log n)/n$ as in this case there are, with high probability, isolated vertices. Any of these vertices keeps its original color forever. In this case, the number of Blue vertices converges with time, and we obtain a bound on the limit in [17].

One can use Theorem 1.5 to derive a "delayed" version (in which Red may need more than 4 days) of Theorem 1.4, by first proving that with high probability, one side gains an advantage of size at least $C\sqrt{n}$ after the first day, for some constant C. This "majority side" then wins with high probability given $p \geq \lambda n^{-1/2}$ (which satisfies the requirement $p \geq (1 + \Omega(1))(\log n)/n$) with λ sufficiently large so that $\lambda C = pC\sqrt{n}$ is large. The detailed argument is in Appendix A.

1.5 Notation

Below is the list of notations used throughout the paper.

- $\mathbf{1}_S$: membership function of the set $S \subset V$, so that $\mathbf{1}_S(v) := \mathbf{1}_{v \in S}$.
- R_t, B_t: Respectively the sets of Red and Blue vertices after day t. (At this point everyone has updated their color t times.)

- $\mathrm{I}_t(u) := \mathbf{1}_{\{u \in R_t\}}$: indicator of the event that u is Red after day t, taking values in $\{0,1\}$.
- $\mathrm{J}_t(u) := 2\mathrm{I}_t(u) - 1$: indicator of the same event, but taking values in $\{-1,1\}$.
- $u \sim v \equiv (u,v) \in E$: Event that u and v are adjacent.
- $N(v) := \{u : u \sim v\}$: The neighborhood of v.
- $W_{uv} := \mathbf{1}_{\{u \sim v\}}$ - Indicator of the adjacency between u and v.
- $\Phi(a,b) := \int_a^b \frac{e^{-x^2/2}}{\sqrt{2\pi}} dx$ and $\Phi(a) := \Phi(-\infty, a)$, $\Phi_0(a) := \Phi(0,a)$. Note that $\Phi(a) = \Phi_0(a) + 1/2$.
- Given a random variable X, we let $\mathrm{supp}(X)$ be the support of X, namely the smallest subset S of $\mathbb{R}$ such that $\mathbf{P}(X \notin S) = 0$. When X is discrete, $\mathrm{supp}(X)$ is simply the set of possible values of X.

1.6 Main Theorem

The main theorem concerns dense graphs, where p is at least a constant. When given appropriate values for the parameters, it implies the "Power of Few" phenomenon in Theorem 1.2.

Theorem 1.6. *Assume p is a real number in $(0,1)$ and n and c are positive integers such that*

$$F(n,p,c) := \sqrt{n}\Phi_0\left(\frac{2pc + \min\{p, 1-p\}}{\sqrt{p(1-p)n}}\right) - .6\frac{1-2p+2p^2}{\sqrt{p(1-p)}} - \frac{.8}{p} > 0. \tag{1.1}$$

Consider the election process on $G \sim G(n,p)$ with $|B_0| \le n/2 - c$. We have

- $\mathbf{P}\left(|B_1| \le \frac{n}{2} - \frac{.8\sqrt{n}}{p}\right) \ge 1 - P_1$, *where*

$$P_1 = P_1(n,p,c) := \frac{.25 + .7\left(1-2p+2p^2\right)^2\left(1+4c^2n^{-2}\right)}{F^2(p,n,c)}.$$

- $\mathbf{P}\left(|B_2| \le .4n \;\middle|\; |B_1| \le \frac{n}{2} - \frac{.8\sqrt{n}}{p}\right) \ge 1 - P_2$, *where*

$$P_2 = P_2(n) := n^{-1}\exp\left(-(.076n - 1.38\sqrt{n})\right).$$

- $\mathbf{P}\left(|B_3| \le \frac{p(n-1)}{3} \;\middle|\; |B_2| \le .4n\right) \ge 1 - P_3$, *where*

$$P_3 = P_3(n,p) := n^{-1}\exp\left(-.02n(p^3 n - 80)\right).$$

- $\mathbf{P}\left(|B_4| = 0 \,\middle|\, |B_3| \le \frac{p(n-1)}{3}\right) \ge 1 - P_4$, *where*

$$P_4 = P_4(n,p) \;:=\; n\exp\left(-\frac{2}{9}p^2(n-1)\right).$$

Consequently, $R_4 = V(G)$ *with probability at least* $1 - (P_1 + P_2 + P_3 + P_4)$.

Note that the condition (1.1) is asymptotically the same as $c = \Omega(p^{-3/2})$.
The proof for this theorem has two main parts corresponding to the next two sections.

1. One day 1, the number of Red and Blue neighbors of each node are binomial with means roughly $n/2 + c$ and $n/2 - c$ respectively. The central limit theorem then implies that most of their masses are concentrated within an interval of length $\Theta(\sqrt{n})$ around their respective expectations. A subinterval of constant length in this interval has $\Theta(n^{-1/2})$ mass. We thus expect that the probability that Reds outnumber Blues in the neighborhood of a given node is $1/2 + \Omega(n^{-1/2})$. Thus, we expect that the number of Red nodes after the first day is $n/2 + \Omega(n^{1/2})$, meaning $n/2 - \Omega(n^{1/2})$ Blue nodes.[1] We use this intuition to find the right bound for $|B_1|$ in Theorem 1.6, and prove it in Section 2.

2. On subsequent days, the coloring depends on the graph so the argument for day 1 is no longer available. We instead show that the number of Blues decreases regardless of the coloring after day 1, as long as Red has $\Omega(\sqrt{n})$ extra nodes. Using Hoeffding bound (Theorem 3.1), we show that for a given coloring, the probability one can find "many" nodes with a majority of neighbors in Blue is small, sufficiently to beat the union bound over all colorings. This bypasses the dependency issue after day 1. The details are in Section 3.

From Theorem 1.6, one can deduce Theorems 1.2 and 1.3 in a few steps.

Proof of Theorem 1.2. Assume Theorem 1.6. Observe that if Equation (1.1) holds for some value of n, then it holds for all larger values of n. Let $n \ge 1000$, $p = 1/2$ and $c = 5$, we have (1.1) satisfied. A routine calculation then shows that

$$P_1(n,p,c) \le .0902, \quad \max\left\{P_2(n), P_3(n,p), P_4(n,p)\right\} < 10^{-10},$$

which implies that $\mathbf{P}(B_4 \ne \varnothing) < .1$ or equivalently that Red wins in the fourth day with probability at least .9 (conditioned on the event $|B_0| \le \frac{n}{2} - c$). ☐

Proof of Theorem 1.3. In this proof, only n and $c = c_n$ can vary. We can assume, without loss of generality, that $c_n \le n/2$. Assuming Theorem 1.6, a routine calculation shows that

[1] While this paper was under review, Sah and Sawhney in a recent paper [14] showed that $|B_1|$ obeys a Gaussian limit law with mean $n/2 - \Theta(c\sqrt{pn})$ and variance $\Theta(n)$ when $n \to \infty$ for $p > (\log n)^{-1/16}$.

$P_2, P_3, P_4 = \mathrm{o}(n^{-2})$ and, so $P_1 + P_2 + P_3 + P_4 = P_1 + \mathrm{o}(n^{-2})$. By Theorem 1.6, we then have

$$\mathbf{P}(R_4 = V(G)) \geq 1 - (P_1 + o(n^{-2})) = 1 - \frac{.25 + .7(1-2p+2p^2)^2\left(1+4c^2n^{-2}\right)}{\left(\sqrt{n}\,\Phi_0\left(\frac{2pc_n + \min\{p,1-p\}}{\sqrt{p(1-p)n}}\right) - \frac{.6(1-2p+2p^2)}{\sqrt{p(1-p)}} - \frac{.8}{p}\right)^2} - o(n^{-2}).$$

We make use of the fact Φ_0 is increasing, while $x \mapsto \Phi_0(x)/x$ is decreasing, which implies $\Phi_0(x) \geq \min\{1, x/y\}\Phi_0(y)\ \forall x, y > 0$. Then:

$$\begin{aligned}\sqrt{n}\Phi_0\left(\frac{2pc_n + \min\{p,1-p\}}{\sqrt{p(1-p)n}}\right) &\geq \sqrt{n}\min\left\{1, \frac{c_n}{\sqrt{(1-p)n}}\right\}\Phi_0\left(2\sqrt{p}\right) \\ &\geq \min\left\{\sqrt{n}, \frac{c_n}{\sqrt{1-p}}\right\}\Phi_0\left(2\sqrt{p}\right) \geq \min\left\{\sqrt{n}, c_n\right\}\Phi_0\left(2\sqrt{p}\right).\end{aligned}$$

When n and c_n are both sufficiently large and p is a constant, we have

$$\min\left\{\sqrt{n}, c_n\right\}\Phi_0\left(2\sqrt{p}\right) \geq 2\left[\frac{.6(1-2p+2p^2)}{\sqrt{p(1-p)}} + \frac{.8}{p}\right]$$

Therefore we can give a lower bound on the denominator of P_1:

$$\sqrt{n}\,\Phi_0\left(\frac{2pc_n + \min\{p,1-p\}}{\sqrt{p(1-p)n}}\right) - \frac{.6(1-2p+2p^2)}{\sqrt{p(1-p)}} - \frac{.8}{p} \geq \frac{\Phi_0\left(2\sqrt{p}\right)}{2}\min\left\{\sqrt{n}, c_n\right\}.$$

Now we use the trivial facts $c_n \leq n/2$ and $1 - 2p + 2p^2 \leq 1$ to give an upper bound on the numerator of P_1:

$$.25 + .7(1-2p+2p^2)^2\left(1+4c^2n^{-2}\right) \leq .25 + .7 \cdot 1 \cdot (1+1) < 2.$$

Combining the above, we get

$$P_1 \leq \frac{2}{\left(\min\{c_n, \sqrt{n}\}\Phi_0(2\sqrt{p})/2\right)^2} = \frac{4\Phi_0(2\sqrt{p})^{-2}}{\min\{\sqrt{n}, c_n\}^2} = 4\Phi_0(2\sqrt{p})^{-2}\max\left\{\frac{1}{n}, \frac{1}{c_n^2}\right\}.$$

The term $o(n^{-2})$ is also absorbed into this bound, since $n^{-2} < n^{-1}$. Therefore we have

$$\mathbf{P}(R_4 = V(G)) \geq 1 - K(p)\max\left\{\frac{1}{n}, \frac{1}{c_n^2}\right\}.$$

which is the desired bound where $K(p)$ can be chosen as $4\Phi_0(2\sqrt{p})^{-2} + 1$. $\square$

1.7 Open questions

Let $\rho(p,k,n)$ be the probability that Red wins if its camp has size $n/2+k$ in the beginning. Theorem 1.2 shows that $\rho(.5,5,n) \geq .9$ (given that n is sufficiently large). In other words, five defectors guarantee Red's victory with overwhelming odds when $p = 1/2$. In fact, we have $\rho(.5,4,n) \geq .77$ by plugging in the same values for ε_1 and ε_2 with $c = 4$ in the proof of Theorem 1.2. We conjecture that one defector already brings a non-trivial advantage.

Conjecture 1.7 (The power of one)**.** *There is a constant $\delta(p) > 0$ for each constant $p > 0$ such that $\rho(p,1,n) \geq 1/2 + \delta(p)$ for all sufficiently large n.* [1]

In the following numerical experiment, we run $T = 10000$ independent trials. In each trial, we fix a set of $N = 10000$ nodes with 5001 Red and 4999 Blue (meaning $c = 1$), generate a graph from $G(N,1/2)$, and simulate the process on the resulting graph. We record the number of wins and the number of days to achieve the win in percentage in Table 1. Among others, we see that Red wins within 3 days with frequency more than .9. The source code for the simulation along with execution instructions can be found online at `https://github.com/thbl2012/majority-dynamics-simulation`.

T	p	Red	Blue	Winner	Last day	Count	Frequency
10^4	1/2	5001	4999	Blue	3	496	4.96 %
10^4	1/2	5001	4999	Blue	4	77	0.77 %
10^4	1/2	5001	4999	Blue	5	3	0.03 %
10^4	1/2	5001	4999	Blue	7	1	0.01 %
10^4	1/2	5001	4999	Red	2	25	0.25 %
10^4	1/2	5001	4999	Red	3	9313	93.13 %
10^4	1/2	5001	4999	Red	4	85	0.85 %

Table 1: Winners and winning days with their frequencies

Imagine that people defect from the Blue camp to the Red camp one by one. The *value* of the i-th defector is defined as $v(p,i,n) := \rho(p,i,n) - \rho(p,i-1,n)$ (where we take $\rho(p,0,n) = 1/2$). It is intuitive to think that the value of each extra defector decreases. (Clearly defector number $n/2$ adds no value.)

Conjecture 1.8 (Values of defectors)**.** *For any fixed p, i, and n large enough, $v(p,i,n) \geq v(p,i+1,n)$.* [2]

It is clear that the Conjecture 1.8 implies Conjecture 1.7, with $\delta = \frac{.4}{5} = .08$, although the simulation results above suggests that δ can be at least .43.

In the next two sections we give the proof of our main result, Theorem 1.6.

[1]Both Conjectures 1.7 and 1.8 are proven for $p > (\log n)^{-1/16}$ in [14]. See footnote 1 a few paragraphs below Theorem 1.6.

[2]See footnote 1 after Conjecture 1.7.

2 Day One

Firstly, let us recall a few terms defined in Theorem 1.6.

$$\begin{aligned} F(n,p,c) &:= \sqrt{n}\Phi_0\left(\frac{2pc+\min\{p,1-p\}}{\sqrt{p(1-p)n}}\right) - .6\frac{1-2p+2p^2}{\sqrt{p(1-p)}} - \frac{.8}{p}, \\ P_1(n,p,c) &:= \frac{.25+.7\left(1-2p+2p^2\right)^2\left(1+4c^2n^{-2}\right)}{F(n,p,c)^2}. \end{aligned}$$

Lemma 2.1. *Given $p \in (0,1)$ and $n, c \in \mathbb{N}$ such that $F(n,p,c) > 0$, The election process on $G \sim G(n,p)$ with $\left|B_0\right| \leq n/2 - c$ satisfies*

$$\mathbf{P}\left(\left|B_1\right| > \frac{n}{2} - .8\frac{\sqrt{n}}{p}\right) \leq P_1(n,p,c).$$

The core of the proof relies on some preliminary results regarding the difference of two binomial random variables, which we discuss next.

2.1 Background on the difference of Binomial Random Variables

The difference of two binomial random variables with the same probability p can be written as a sum of independent random variables, each of which is either a $\text{Bin}(1,p)$ variable or minus of one. A natural way to bound this sum is done via a Berry–Esseen normal approximation.

Theorem 2.2 (Berry–Esseen)**.** *Let n be any positive integer. If $X_1, X_2, X_3, \ldots, X_n$ are random variables with zero means, variances $\sigma_1^2, \sigma_2^2, \cdots, \sigma_n^2 > 0$, and absolute third moments $\mathbf{E}\left[|X_i|^3\right] = \rho_i < \infty$, we have:*

$$\sup_{x\in\mathbb{R}}\left|\mathbf{P}\left(\sum_{i=1}^{n} X_i \leq x\right) - \Phi\left(\frac{x}{\sigma_X}\right)\right| \leq C_0 \cdot \frac{\sum_{i=1}^{n}\rho_i}{\sigma_X^3},$$

where $\sigma_X = \left(\sum_{i=1}^{n}\sigma_i^2\right)^{1/2}$ and C_0 is a constant.

The original proof by Esseen [7] yielded $C_0 = 7.59$, and this constant has been improved a number of times. The latest work by Shevtsova [15] achieved $C_0 = .56$, which will be used for the rest of the paper. A direct application of this theorem gives the following lemma.

Lemma 2.3. *For $p \in (0,1)$, $\sigma = \sqrt{p(1-p)}$ and $n_1, n_2 \in \mathbb{N}$ such that $n_1 > n_2$. let $Y_1 \sim Bin(n_1,p)$, $Y_2 \sim Bin(n_2,p)$ be independent random variables. Then for any $d \in \mathbb{R}$,*

$$\mathbf{P}\left(Y_1 > Y_2 + d\right) \geq \frac{1}{2} + \Phi_0\left(\frac{p(n_1-n_2)-d}{\sqrt{p(1-p)(n_1+n_2)}}\right) - \frac{.56\left(1-2p+2p^2\right)}{\sqrt{p(1-p)(n_1+n_2)}}.$$

Proof. Let $X = Y_1 - Y_2$. By definition, we have $X = X_1 + X_2 + X_3 + \cdots + X_{n_1+n_2}$, where all the X_i are independent and either X_i or $-X_i$ is $\mathrm{Bin}(1, p)$. Then $\mathbf{E}[X] = \sum_i \mathbf{E}[X_i] = p(n_1 - n_2)$. For all i,

$$\mathbf{Var}[X_i] = p(1-p) \text{ and } \mathbf{E}\left[|X_i - \mathbf{E}[X_i]|^3\right] = p(1-p)^3 + (1-p)p^3 = p(1-p)(1-2p+2p^2).$$

Applying Theorem 2.2, we have

$$\begin{aligned}
\mathbf{P}(Y_1 \le Y_2 + d) &= \mathbf{P}(X - \mathbf{E}[X] \le d - p(n_1 - n_2)) \\
&\le \Phi\left(\frac{d - p(n_1-n_2)}{\sqrt{\mathbf{Var}[X]}}\right) + .56\frac{\sum_i \mathbf{E}\left[|X_i - \mathbf{E}[X_i]|^3\right]}{\mathbf{Var}[X]^{3/2}} \\
&= \Phi\left(\frac{d - p(n_1-n_2)}{\sqrt{p(1-p)(n_1+n_2)}}\right) + .56\frac{p(1-p)(1-2p+2p^2)(n_1+n_2)}{(p(1-p)(n_1+n_2))^{3/2}} \\
&= \frac{1}{2} - \Phi_0\left(\frac{p(n_1-n_2) - d}{\sqrt{p(1-p)(n_1+n_2)}}\right) + \frac{.56(1-2p+2p^2)}{\sqrt{p(1-p)(n_1+n_2)}},
\end{aligned}$$

and the claim follows by taking the complement event. □

Lemma 2.4. *Let $p \in (0,1)$ be a constant, and $X_1 \sim Bin(n_1, p)$ and $X_2 \sim Bin(n_2, p)$ be independent r.v.s. Then for any integer d,*

$$\mathbf{P}(X_1 = X_2 + d) \le \frac{1.12\left(1-2p+2p^2\right)}{\sqrt{p(1-p)(n_1+n_2)}}.$$

Proof. Let $n = n_1 + n_2$ and $\mu = \mathbf{E}[X_1] - \mathbf{E}[X_2] = p(n_1 - n_2)$. Fix $\varepsilon \in (0,1)$, by the same computations in Lemma 2.3, we have

$$\begin{aligned}
\mathbf{P}(X_1 - X_2 \le d - \varepsilon) &\ge \Phi\left(\frac{d-\mu-\varepsilon}{\sqrt{p(1-p)n}}\right) - \frac{.56\left(1-2p+2p^2\right)}{\sqrt{p(1-p)n}}, \\
\mathbf{P}(X_1 - X_2 < d + \varepsilon) &\le \Phi\left(\frac{d-\mu+\varepsilon}{\sqrt{p(1-p)n}}\right) + \frac{.56\left(1-2p-2p^2\right)}{\sqrt{p(1-p)n}}.
\end{aligned}$$

It follows that

$$\begin{aligned}
\mathbf{P}(X_1 = X_2 + d) &\le \mathbf{P}(d - \varepsilon < X_1 - X_2 < d + \varepsilon) \\
&\le \Phi\left(\frac{d-\mu+\varepsilon}{\sqrt{p(1-p)n}}\right) - \Phi\left(\frac{d-\mu-\varepsilon}{\sqrt{p(1-p)n}}\right) + \frac{1.12\left(1-2p+2p^2\right)}{\sqrt{p(1-p)n}}.
\end{aligned}$$

Letting $\varepsilon \to 0$, we obtain the desired claim. □

Proof of Lemma 2.1

Recall that $|R_1| = n - |B_1|$. Our goal is to give an upper bound on the probability that $|R_1| < n/2 + d\sqrt{n}$ for any given term d. Recall that $|R_1| = \sum_{v\in V} \mathrm{I}_1(v)$, where $\mathrm{I}_1(v)$ is 1 if v is Red after Day 1 and 0 otherwise. We have: Since the indicators are not independent, a natural choice for bounding their sum is to use Chebysev's inequality. We proceed in two steps:

1. Give a lower bound on $\mathbf{E}\left[|R_1|\right]$ by lowerbounding each term $\mathbf{E}\left[\mathrm{I}_1(v)\right]$.

2. Give an upper bound on $\mathbf{Var}\left[|R_1|\right]$ by upperbounding each $\mathbf{Var}\left[\mathrm{I}_1(v)\right]$ and $\mathbf{Cov}\left[\mathrm{I}_1(v), \mathrm{I}_1(v')\right]$.

Let $\sigma := \sqrt{p(1-p)}$, which appears in most equations in this proof. Note that $1-2p+2p^2 = 1-2\sigma^2$.

Claim 2.5. $\mathbf{E}\left[|R_1|\right] \geq \dfrac{n}{2} + F_1(n,p,c)\sqrt{n}$, *where*

$$F_1(n,p,c) = \sqrt{n}\Phi_0\left(\frac{2pc + \min\{p, 1-p\}}{\sigma\sqrt{n}}\right) - .6\frac{1-2\sigma^2}{\sigma} = F(n,p,c) + \frac{.8\sqrt{n}}{p}.$$

Proof. For $v \in V$ and $S \subset V$, let $d_S(v)$ be the number of neighbors v has in S. By the majority and tie-breaking rules , we have for each $v \in V$,

$$v \in R_1 \iff d_{R_0}(v) > d_{B_0}(v) - \mathrm{I}_0(v). \tag{2.1}$$

Note that $d_{R_0}(v) \sim \mathrm{Bin}\left(|R_0| - \mathrm{I}_0(v), p\right)$ and $d_{B_0}(v) \sim \mathrm{Bin}\left(|B_0| + \mathrm{I}_0(v) - 1, p\right)$. By Lemma 2.3, we have:

$$\begin{aligned}
&\mathbf{E}\left[\mathrm{I}_1(v)\right] = \mathbf{P}\left(v \in R_1\right) = \mathbf{P}\left(d_{R_0}(v) > d_{B_0}(v) - \mathrm{I}_0(v)\right)\\
&\geq \frac{1}{2} + \Phi_0\left(\frac{p\left(|R_0| - |B_0| + 1 - 2\mathrm{I}_0(v)\right) + \mathrm{I}_0(v)}{\sigma\sqrt{n-1}}\right) - \frac{.56\left(1-2\sigma^2\right)}{\sigma\sqrt{n-1}}\\
&= \frac{1}{2} + \Phi_0\left(\frac{2pc + p_v}{\sigma\sqrt{n-1}}\right) - \frac{.56\left(1-2\sigma^2\right)}{\sigma\sqrt{n-1}} \geq \frac{1}{2} + \Phi_0\left(\frac{2pc + \min\{p, 1-p\}}{\sigma\sqrt{n}}\right) - \frac{.6\left(1-2\sigma^2\right)}{\sigma\sqrt{n1}},
\end{aligned}$$

where $p_v := p\left(1 - \mathrm{I}_0(v)\right) + (1-p)\mathrm{I}_0(v) \geq \min\{p, 1-p\}$, hence the last inequality. Summing over all $v \in V$, we get

$$\mathbf{E}\left[|R_1|\right] \geq \frac{n}{2} + \left[\sqrt{n}\Phi_0\left(\frac{2pc + \min\{p, 1-p\}}{\sigma\sqrt{n}}\right) - \frac{.6\left(1-2\sigma^2\right)}{\sqrt{p(1-p)}}\right]\sqrt{n}.$$

The proof is complete. ◻

Claim 2.6. $\mathbf{Var}\left[|R_1|\right] \leq .25n + .7\left(1-2\sigma^2\right)^2\left(n + 4c^2n^{-1}\right).$

Proof. We first have:

$$\mathbf{Var}\left[\left|R_1\right|\right] = \sum_{v \in V} \mathbf{Var}\left[\mathrm{I}_1(v)\right] + 2 \sum_{v_1 \neq v_2} \mathbf{Cov}\left[\mathrm{I}_1(v_1), \mathrm{I}_1(v_2)\right].$$

The variances $\mathbf{Var}[\mathrm{I}_1(v)]$ are easy due to $\mathrm{I}_1(v)$ being a Bernoulli variable:

$$\mathbf{Var}\left[\mathrm{I}_1(v)\right] = \mathbf{E}\left[\mathrm{I}_1(v)\right]\left(1 - \mathbf{E}\left[\mathrm{I}_1(v)\right]\right) = .25 - \left(\mathbf{E}\left[\mathrm{I}_1(v)\right] - .5\right)^2 \leq .25. \tag{2.2}$$

Bounding the covariance $\mathbf{Cov}[\mathrm{I}_1(v_1), \mathrm{I}_1(v_2)]$ for two distinct vertices v_1, v_2 requires a bit more care, as the indicators are dependent. By definition

$$\mathbf{Cov}\left[\mathrm{I}_1(v_1), \mathrm{I}_1(v_2)\right] \;=\; \mathbf{P}\left(v_1, v_2 \in R_1\right) - \mathbf{P}\left(v_1 \in R_1\right) \mathbf{P}\left(v_2 \in R_1\right).$$

Consider the event $\{v_1, v_2 \in R_1\}$; $\mathbf{P}(v_1, v_2 \in R_1)$ can be written as

$$\mathbf{P}\left(v_1, v_2 \in R_1 | v_1 \sim v_2\right) \mathbf{P}(v_1 \sim v_2) + \;\mathbf{P}\left(v_1, v_2 \in R_1 | v_1 \not\sim v_2\right) \mathbf{P}(v_1 \not\sim v_2).$$

Notice that after we reveal the adjacency between v_1 and v_2, the remaining vertices in the neighborhoods of v_1 and v_2 are independent. Letting $a_i = \mathbf{P}(v_i \in R_1 \mid v_1 \sim v_2)$, $b_i = \mathbf{P}(v_i \in R_1 \mid v_1 \not\sim v_2)$, we have

$$\mathbf{P}\left(v_1, v_2 \in R_1\right) = p a_1 a_2 + (1-p) b_1 b_2.$$

Splitting up the other two events similarly gives $\mathbf{P}(v_i \in R_1) = p a_i + (1-p) b_i$ for $i = 1, 2$. Putting the above together, we can write $\mathbf{Cov}[\mathrm{I}_1(v_1), \mathrm{I}_1(v_2)]$ as

$$\begin{aligned} &p a_1 a_2 + (1-p) b_2 b_2 - \left(p a_1 + (1-p) b_1\right)\left(p a_2 + (1-p) b_2\right) \\ &= \; p(1-p)(a_1 - b_1)(a_2 - b_2) = \sigma^2 (a_1 - b_1)(a_2 - b_2). \end{aligned} \tag{2.3}$$

To bound $a_i - b_i$, we use logical reasoning to express it as something separate from the current context. We start with $i = 1$. The analysis for $a_2 - b_2$ is analogous. Assume $v_1 \sim v_2$ for now, then

$$a_1 = \mathbf{P}\left(d_{R_0}(v_1) \geq d_{B_0}(v_1) - \mathrm{I}_0(v_1) + 1\right),$$

by Eq. (2.1). Define two binomial random variables:

$$\begin{aligned} X &= d_{R_0}(v_1) - \mathrm{I}_0(v_2) = \left|N(v_1) \cap R_0 \setminus \{v_1, v_2\}\right| \sim \mathrm{Bin}\Big(\left|R_0\right| - \mathrm{I}_0(v_1) - \mathrm{I}_0(v_2), p\Big), \\ Y &= d_{B_0}(v_1) + \mathrm{I}_0(v_2) - 1 = \left|N(v_1) \cap B_0 \setminus \{v_1, v_2\}\right| \sim \mathrm{Bin}\Big(\left|B_0\right| + \mathrm{I}_0(v_1) + \mathrm{I}_0(v_2) - 2, p\Big). \end{aligned}$$

We have

$$a_1 = \mathbf{P}\left(X + \mathrm{I}_0(v_2) + \mathrm{I}_0(v_1) \geq Y + 2 - \mathrm{I}_0(v_2)\right) = \mathbf{P}\left(X - Y \geq 1 - \mathrm{I}_0(v_1) - \mathrm{J}_0(v_2)\right).$$

Similarly, when conditioning on $v_1 \not\sim v_2$, we get $b_1 = \mathbf{P}\left(X - Y \geq 1 - \mathrm{I}_0(v_1)\right)$.

Let us perform a case analysis on the initial color of v_2.

$$v_2 \in R_0 \;\Rightarrow\; a_1 - b_1 = \mathbf{P}(X - Y \geq -\mathrm{I}_0(v_1)) - \mathbf{P}(X - Y \geq 1 - \mathrm{I}_0(v_1)) = \mathbf{P}(X - Y = -\mathrm{I}_0(v_1)).$$
$$v_2 \in B_0 \;\Rightarrow\; a_1 - b_1 = \mathbf{P}(X - Y \geq 2 - \mathrm{I}_0(v_1)) - \mathbf{P}(X - Y \geq 1 - \mathrm{I}_0(v_1)) = -\mathbf{P}(X - Y = 1 - \mathrm{I}_0(v_1)).$$

Both cases give us

$$a_1 - b_1 = \mathrm{J}_0(v_2)\mathbf{P}\Big(X - Y = 1 - \mathrm{I}_0(v_1) - \mathrm{I}_0(v_2)\Big),$$

where $X \sim \mathrm{Bin}\Big(|R_0| - \mathrm{I}_0(v_1) - \mathrm{I}_0(v_2), p\Big)$ and $Y \sim \mathrm{Bin}\Big(|B_0| + \mathrm{I}_0(v_1) + \mathrm{I}_0(v_2) - 2, p\Big)$.
We apply the same analysis for a_2 and b_2 and use Eq. (2.3) to get

$$\mathbf{Cov}\left[\mathrm{I}_1(v_1), \mathrm{I}_1(v_2)\right] = \sigma^2 \mathrm{J}_0(v_1)\mathrm{J}_0(v_2)\mathbf{P}\Big(X - Y = 1 - \mathrm{I}_0(v_1) - \mathrm{I}_0(v_2)\Big)^2.$$

If the initial colors of v_1 and v_2 are different $\mathbf{Cov}\left[\mathrm{I}_1(v_1), \mathrm{I}_1(v_2)\right] \leq 0$, so we can exclude them from the upper bound. When v_1 and v_2 are of the same initial color, by Lemma 2.4, we have

$$\mathbf{Cov}\left[\mathrm{I}_1(v_1), \mathrm{I}_1(v_2)\right] \leq \sigma^2 \left(\frac{1.12\,(1 - 2\sigma^2)}{\sigma\sqrt{n-2}}\right)^2 \leq 1.4\frac{\left(1 - 2\sigma^2\right)^2}{n}.$$

Hence

$$\begin{aligned}\sum_{v_1 \neq v_2} \mathbf{Cov}\left[\mathrm{I}_1(v_1), \mathrm{I}_1(v_2)\right] &\leq \left[\binom{|R_0|}{2} + \binom{|B_0|}{2}\right] \cdot 1.4 \cdot \frac{\left(1 - 2\sigma^2\right)^2}{n} \\ &= \frac{1.4}{n}\left[\frac{n^2}{4} + c^2 - \frac{n}{2}\right]\left(1 - 2\sigma^2\right)^2 \leq .35\left(1 - 2\sigma^2\right)^2\left(n + \frac{4c^2}{n}\right).\end{aligned}$$

Equations (2.2) and (2.1) together yield

$$\mathbf{Var}\left[|R_1|\right] \leq .25n + .7\left(1 - 2\sigma^2\right)^2\left(n + 4c^2 n^{-1}\right). \tag{2.4}$$

The proof is complete. □

From Claims 2.5 and 2.6, Chebyshev's inequality gives

$$\begin{aligned}\mathbf{P}\left(|B_1| > \frac{n}{2} - \frac{.8}{p}\sqrt{n}\right) &= \mathbf{P}\left(|R_1| < \frac{n}{2} + \frac{.8}{p}\sqrt{n}\right) \\ \leq \frac{\mathbf{Var}\left[|R_1|\right]}{\left(\mathbf{E}\left[|R_1|\right] - \frac{n}{2} - \frac{.8}{p}\sqrt{n}\right)^2} &\leq \frac{.25n + .7\left(1 - 2\sigma^2\right)^2\left(n + 4c^2 n^{-1}\right)}{n\left(\sqrt{n}\Phi_0\left(\frac{2pc + \min\{p, 1-p\}}{\sigma\sqrt{n}}\right) - .6\frac{1 - 2\sigma^2}{\sqrt{p(1-p)}} - \frac{.8}{p}\right)^2}.\end{aligned}$$

Dividing both the nominator and denominator by n and substituting $\sigma = \sqrt{p(1-p)}$, we get back $P_1(n, p, c)$ and finish the proof of Lemma 2.1. art one of Theorem 1.6 is complete.

3 Day Two and after

Next, we analyze the situation after the first day. As mentioned in Section 1.6, the previous argument is unusable due to the loss of independence from day 2 onwards. Instead, we use "shrinking arguments" to argue that it is likely for the Blue camp to repeatedly shrink to empty, regardless of the choice of its members, due to the structure of G. The core of our shrinking argument is *Hoeffding's inequality*, a classical result that gives exponentially small probability tails for sums of independent random variables.

Theorem 3.1 (Hoeffding's inequality). *Let $\{X_i\}_{i=1}^n$ be independent random variables and $\{a_i\}_{i=1}^n$, $\{b_i\}_{i=1}^n$ be real numbers such that for all $i = \overline{1, n}$, supp$(X) \subseteq [a_i, b_i]$. Then for $X = X_1 + X_2 + \cdots + X_n$, we have*

$$\max\Big\{\mathbf{P}\left(X \geq \mathbf{E}\left[X\right] + t\right),\ \mathbf{P}\left(X \leq \mathbf{E}\left[X\right] - t\right)\Big\} \leq \exp\left(-\frac{2t^2}{\sum_{i=1}^n (b_i - a_i)^2}\right).$$

The proof of Hoeffding's inequality is available in several graduate level probability textbooks, e. g., [18]. The original proof given by Hoeffding appeared in [9].

Before discussing our shrinking arguments, we quickly record a useful lemma to simplify binomial coefficients that may appear in union bounds.

Lemma 3.2. *For any integers $n \geq 1$ and $0 \leq k \leq n$, $\binom{n}{k} < \frac{2^n}{\sqrt{n}}$.*

Proof. This is well known. Observing that $\binom{n}{k} \leq \binom{n}{\lfloor n/2 \rfloor}$, the case of even n is stated in [11, 2.6.2 Proposition]. The case of odd $n = 2m - 1$ follows immediately:

$$\binom{2m-1}{m-1} = \frac{1}{2}\binom{2m}{m} < \frac{1}{2}\frac{2^{2m}}{\sqrt{2m}} = \frac{2^{2m-1}}{\sqrt{2m}} < \frac{2^{2m-1}}{\sqrt{2m-1}}. \qquad \square$$

A simple yet useful shrinking argument is that, in the $G(n, p)$ model, it is with high probability that all vertices in G have many neighbors, so a small enough Blue camp will not be able to influence anyone by a majority, thus inevitably vanishes the next day.

Lemma 3.3 (Part 4 of Theorem 1.6)**.** *For $p \in (0, 1)$ and $n \in \mathbb{N}_{>1}$, with probability at least*

$$1 - n\exp\left(-\frac{2}{9}p^2(n-1)\right) = 1 - P_4(n, p),$$

G is such that all vertices have more than $\frac{2}{3}p(n-1)$ neighbors, thus any choice of the Blue camp of at most $\frac{1}{3}p(n-1)$ members vanishes the next day.

Proof. In a $G(n, p)$ graph, $d(v)$ is a sum of $(n-1)$ Bin$(1, p)$ random variables, so Theorem 3.1 implies that for any $u \in V$,

$$\mathbf{P}\left(d(u) \leq \frac{2}{3}p(n-1)\right) \leq \mathbf{P}\left(d(u) - \mathbf{E}\left[d(u)\right] \leq -\frac{1}{3}p(n-1)\right) \leq \exp\left(-\frac{2}{9}p^2(n-1)\right).$$

By a union bound, the probability that all vertices have more than $\frac{2}{3}p(n-1)$ neighbors is at least $1 - n\exp\left(-\frac{2}{9}p^2(n-1)\right) = 1 - P_4(n,p)$. Given this, a Blue camp of size $\frac{1}{3}p(n-1)$ surely vanishes the next day since it cannot form a majority in the neighborhood of any vertex. The result then follows. □

This simple lemma completes the fourth part of Theorem 1.6. The arguments for Parts 2 and 3 require slightly more complexity, but both come directly from the following lemma.

Lemma 3.4. *Let $p \in (0,1)$, $n, n_0 \in \mathbb{N}$, $n_0 < \frac{n}{2}$. Then for all $m \in \mathbb{N}, m \leq n$, with probability at least*

$$1 - \frac{4^n}{n}\exp\left(-\frac{2p^2(n-2n_0-1)^2 m}{n+m-2}\right),$$

G is such that any choice of the Blue camp of at most n_0 members shrinks to below m in the next day.

Proof of Lemma 3.4. Consider a subset S of V with m elements. We first bound the probability that S entirely turns Blue the next day. Let (R,B) be the initial coloring with $|B| = n_0 < n/2$. Similarly to the usual J_t, let $\mathrm{J}(v) = 1$ if $v \in R$ and -1 otherwise. For each $v \in V$, let $\mathrm{dif}(v) := |N(v) \cap R| - |N(v) \cap B| = \sum_{u \in V} \mathrm{J}(u) W_{uv}$, and let $\mathrm{dif}(S) := \sum_{v \in S} \mathrm{dif}(v)$. We break down $\mathrm{dif}(S)$ as follows.

$$\mathrm{dif}(S) = \sum_{v \in V}\left[\mathbf{1}_S(v)\sum_{u \in V}\mathrm{J}(u)W_{uv}\right] = \sum_{u \neq v}\Big[\mathrm{J}(u)\mathbf{1}_S(v) + \mathrm{J}(v)\mathbf{1}_S(u)\Big]W_{uv}. \tag{3.1}$$

This is now a sum of independent variables, so we can apply Theorem 3.1.

$$\mathbf{P}\left(S \subseteq B_1 \mid B_0\right) \;\leq\; \mathbf{P}(\mathrm{dif}(S) \leq 0) \;\leq\; \exp\left[-\frac{\mathbf{E}\left[\mathrm{dif}(S)\right]^2}{\sum_{u \neq v}\left(\sup\mathrm{supp}(X_{uv}W_{uv}) - \inf\mathrm{supp}(X_{uv}W_{uv})\right)^2}\right], \tag{3.2}$$

where $X_{uv} := \mathrm{J}(u)\mathbf{1}_S(v) + \mathrm{J}(v)\mathbf{1}_S(u)$. The following table sums up respective values of X_{uv} for (u,v) among the sets $R \cap S$, $R \setminus S$, $B \cap S$ and $B \setminus S$.

<table>
<tr><th>$u \setminus v$</th><th>$R \cap S$</th><th>$B \cap S$</th><th>$R \cap S^c$</th><th>$B \cap S^c$</th></tr>
<tr><td>$R \cap S$</td><td>2</td><td>0</td><td rowspan="2">1</td><td rowspan="2">−1</td></tr>
<tr><td>$B \cap S$</td><td>0</td><td>−2</td></tr>
<tr><td>$R \cap S^c$</td><td colspan="2">1</td><td colspan="2" rowspan="2">0</td></tr>
<tr><td>$B \cap S^c$</td><td colspan="2">−1</td></tr>
</table>

(3.3)

Substituting back into Equation (3.1), we get

$$\mathrm{dif}(S) \;=\; \sum_{\{u,v\} \subset S \cap R}(2W_{uv}) - \sum_{\{u,v\} \subset S \cap B}(2W_{uv}) + \sum_{u \in S}\sum_{v \in R \setminus S} W_{uv} - \sum_{u \in S}\sum_{v \in B \setminus S} W_{uv}.$$

For convenience in the following computations, let $m_R := |S \cap R|$ and $m_B := |S \cap B|$.

Note that $\left|B \setminus S\right| = n_0 - m_B$ and $\left|R \setminus S\right| = n - n_0 - m_R$. Let W be a $\mathrm{Bin}(1, p)$ random variable. Observe that S is a sum of $\binom{m_R}{2} = \frac{m_R(m_R-1)}{2}$ copies of $2W$, $\binom{m_B}{2} = \frac{m_B(m_B-1)}{2}$ copies of $(-2W)$, $m(n - n_0 - m_R)$ copies of W, and $m(n_0 - m_B)$ copies of $(-W)$, all independent. Therefore

$$\begin{aligned}
\mathbf{E}\left[\mathrm{dif}(S)\right] &= p\left[2 \cdot \frac{m_R(m_R-1)}{2} - 2 \cdot \frac{m_B(m_B-1)}{2} + m(n - n_0 - m_R) - m(n_0 - m_B)\right] \\
&= p\left[m_R^2 - m_B^2 - m_R + m_B + m(n - 2n_0 - m_R + m_B)\right] \\
&= p\left[(m_R - m_B)(m_R + m_B) - m(m_R - m_B) + m(n - 2n_0) - m_R + m_B)\right] \\
&= p\left[m(m_R - m_B) - m(m_R - m_B) + m(n - 2n_0) - (m - m_B) + m_B\right] \\
&= p\left[m(n - 2n_0) - m + 2m_B\right] \geq p\left[m(n - 2n_0) - m\right] = pm(n - 2n_0 - 1).
\end{aligned}$$

Moreover, $\mathrm{supp}(W_{uv}) = 0, 1$, so $\mathrm{supp}(X_{uv}W_{uv}) = \{0, X_{uv}\}$, thus

$$\sup \mathrm{supp}(X_{uv}W_{uv}) - \inf \mathrm{supp}(X_{uv}W_{uv}) = |X_{uv}| \text{ for all } u \neq v.$$

Therefore the denominator of the exponent in Equation (3.2) is

$$\begin{aligned}
&\sum_{u \neq v} \left(\sup \mathrm{supp}(X_{uv}W_{uv}) - \inf \mathrm{supp}(X_{uv}W_{uv})\right)^2 = \sum_{u \neq v} X_{uv}^2 \\
&= \sum_{\{u,v\} \subset S \cap R} 2^2 + \sum_{\{u,v\} \subset S \cap B} (-2)^2 + \sum_{u \in S} \sum_{v \in R \setminus S} 1^2 + \sum_{u \in S} \sum_{v \in B \setminus S} (-1)^2 \\
&= 4 \cdot \frac{m_R(m_R-1)}{2} + 4 \cdot \frac{m_B(m_B-1)}{2} + m(n - n_0 - m_R) + m(n_0 - m_B) \\
&= 2m_R^2 + 2m_B^2 - 2m_R - 2m_B + m(n - m_R - m_B) = 2(m_R + m_B)^2 - 4m_R m_B - 2m + m(n - m) \\
&\leq 2(m_R + m_B)^2 - 2m + m(n - m) = 2m^2 - 2m + m(n - m) = m(n + m - 2).
\end{aligned}$$

Substituting this bound into Equation (3.2), we get

$$\mathbf{P}(S \subseteq B_1 \mid B_0 = B) \leq \exp\left[-\frac{2\left(pm(n - 2n_0 - 1)\right)^2}{m(n + m - 2)}\right] = \exp\left[-\frac{2p^2(n - 2n_0 - 1)^2 m}{n + m - 2}\right].$$

Applying a double union bound over choices of S and B, noting that by Lemma 3.2, the number of choices $\binom{n}{n_0}\binom{n}{m}$ is at most $4^n/n$, so we have

$$\mathbf{P}\left(\exists B_0 \in \binom{V}{n_0}, \exists S \in \binom{V}{m}, S \subset B_1\right) \leq \frac{4^n}{n} \exp\left(-\frac{2p^2(n - 2n_0 - 1)^2 m}{n + m - 2}\right).$$

Taking the complement event, we get the desired result. ◻

Lemma 3.4 is sufficient to prove parts 2 and 3 of Theorem 1.6. The following lemmas are direct corollaries of Lemma 3.4.

Lemma 3.5 (Part 2 of Theorem 1.6). *In the election process on $G \sim G(n, p)$, with probability at least*

$$1 - n^{-1} \exp(-.025n) = 1 - P_2(n), \tag{3.4}$$

G is such that any choice of the Blue camp of at most $n/2 - (.8/p)\sqrt{n} = n_1^B$ members shrinks to size at most $.4n = n_2^B$ the next day.

Proof of Lemma 3.5. Let $n_2 := \lfloor n/2 - (.8/p)\sqrt{n} \rfloor$ and $m := \lceil .4n \rceil$. By Lemma 3.4, G is such that every Blue set of at most n_2 vertices shrinks to size $m - 1$ with probability at least

$$1 - \frac{4^n}{n} \exp\left(-\frac{2p^2(n - 2n_2 - 1)^2 m}{n + m - 2}\right) = 1 - \frac{1}{n} \exp\left[-\left(\frac{2p^2(n - 2n_2 - 1)^2 m}{n + m - 2} - 2n \log 2\right)\right].$$

We have

$$\frac{m}{n + m - 2} \geq \frac{m}{n + m} \geq 2/7, \quad \text{and } n - 2n_2 - 1 \geq \frac{1.6\sqrt{n}}{p} - 1.$$

Therefore can bound the exponent of the RHS of Equation (3) as follows:

$$\begin{aligned} \frac{2p^2(n - 2n_2 - 1)^2 m}{n + m - 2} - 2n \log 2 \; &\geq \; \frac{4}{7} p^2 \left(\frac{1.6\sqrt{n}}{p} - 1\right) - 2n \log 2 \\ = \left(\frac{10.24}{7} - 2 \log 2\right) n &- \frac{12.8}{7} p\sqrt{n} + \frac{4}{7} p^2 \; \geq \; .076n - 1.83\sqrt{n}. \end{aligned}$$

The proof is complete. □

Lemma 3.6 (Part 3 of Theorem 1.6). *In the election process on $G \sim G(n, p)$, with probability at least*

$$1 - n^{-1} \exp\left(-.02n(p^3 n - 80)\right) = 1 - P_3(n, p),$$

G is such that any choice of the Blue camp with at most $.4n$ members shrinks to at most $\frac{1}{3}p(n - 1)$ members the next day.

Proof of Lemma 3.6. Let $n_2 := \lfloor .4n \rfloor$ and $m := \lceil \frac{1}{3} p(n - 1) \rceil$. By Lemma 3.4, G satisfies that every Blue set of at most n_2 vertices shrinks to size $m - 1$ with probability at least

$$1 - \frac{4^n}{n} \exp\left(-\frac{2p^2(n - 2n_2 - 1)^2 m}{n + m - 2}\right) = 1 - \frac{1}{n} \exp\left[-\left(\frac{2p^2(n - 2n_2 - 1)^2 m}{n + m - 2} - 2n \log 2\right)\right]. \tag{3.5}$$

Since $n_2 \leq .4n$, we have $(n - 2n_2 - 1)^2 \geq (.2n - 1)^2$. Furthermore, $m \geq \frac{1}{3} p(n - 1)$ so

$$\frac{m}{n + m - 2} \geq \frac{p(n - 1)/3}{n + p(n - 1)/3 - 1} = \frac{p}{3 + p} \geq \frac{p}{4}.$$

Therefore we can bound the exponent in the RHS of Equation (3.5) as follows:

$$\begin{aligned}
&\frac{2p^2(n-2n_2-1)^2m}{n+m-2} - 2n\log 2 \;\geq\; \frac{2p^2(.2n-1)^2m}{n+m-2} - 2n\log 2 \;\geq\; \frac{p^3(.2n-1)^2}{2} - 2n\log 2\\
&= p^3\left(.02n^2 - .2n + \frac{1}{2}\right) - 2n\log 2 \;\geq\; .02p^3n^2 - (.2p^3 + 2\log 2)n + \frac{p^3}{2}\\
&= .02n(p^3n - 10p^3 - 100\log 2) \;\geq\; .02n(p^3n - 80),
\end{aligned}$$

where the last inequality is due to $10p^3 + 100\log 2 \leq 10 + 100\log 2 < 80$. The result follows. □

These lemmas and Lemma 3.3, together with Lemma 2.1 form the complete "chain of shrinking" for the number of Blue vertices to reach 0 in four days, hence wrapping up the proof of Theorem 1.6.

4 Conclusion

The majority dynamics scheme on a network is a process where each of the n individuals is assigned an initial color, which changes daily to match the majority among their neighbors. Our main result, Theorem 1.6, when the network is a $G(n,p)$ random graph, yields an explicit lower bound based on n, p and c for the probability that the side with the initial majority wins. It has two important implications. The first is a surprising phenomenon, which we call the *Power of Few* phenomenon (Theorem 1.2), which shows that when $p = 1/2$ and $\varepsilon = .1$, c can be set to just 5, meaning five extra people is all it takes to win a large election with overwhelming odds. The second is an asymptotic dependency between the ε, n and c (Theorem 1.3), which shows that for any fixed p, there is a constant $K(p)$ such that choosing n and c both large enough so that $K(p)\max\{n^{-1}, c^{-2}\} < \varepsilon$ will ensure that the winning probability is at least $1-\varepsilon$.

Although the results in this paper only apply to dense $G(n,p)$ graphs, we do cover sparse graphs in a separate in-progress paper [17], where we obtain the Power of Few phenomenon for $p = \Omega((\log n)/n)$, and discuss the end result (other than a win) for lower values of p. We nevertheless mention one of the main results proved in the upcoming paper (Theorem 1.5), and use it to prove the main theorem of the paper [8] by Fountoulakis, Kang and Makai in Appendix A.

A Proof of Fountoulakis et al's Theorem from Theorem 1.5

In this Appendix we give a proof of the main theorem by Fountoulakis et al in [8] (Theorem 1.4) using the main theorem in our upcoming paper (Theorem 1.5).

Proof. Assume Theorem 1.5. Let R_0 and B_0 respectively be the initial Red and Blue camps. Fix a constant $0 < c' \leq \varepsilon/6$. $|R_0| \sim \mathrm{Bin}(n, 1/2)$ since it is a sum of $\mathrm{Bin}(1, 1/2)$ variables. An application of the Berry–Esseen theorem (Theorem 2.2; with .56 = .56) implies that

$$\mathbf{P}\left(|R_0| - \frac{n}{2} \leq c'\sqrt{n}\right) \leq \Phi(2c') + \frac{.56}{\sqrt{n}}, \quad \text{and } \mathbf{P}\left(|R_0| - \frac{n}{2} \leq -c'\sqrt{n}\right) \geq \Phi(-2c') - \frac{.56}{\sqrt{n}},$$

Thus

$$\mathbf{P}\left(\left|\left|R_0\right| - \frac{n}{2}\right| \le c'\sqrt{n}\right) \le \left(\Phi(2c') + \frac{.56}{\sqrt{n}}\right) - \left(\Phi(-2c') - \frac{.56}{\sqrt{n}}\right)$$
$$\le \Phi(-2c', 2c') + \frac{20.56}{\sqrt{n}} \le \frac{4c'}{\sqrt{2\pi}} + \frac{20.56}{\sqrt{n}} \le \frac{\varepsilon}{3} + \frac{20.56}{\sqrt{n}} \le \varepsilon/2,$$

for sufficiently large n.

On the other hand, if $\left|\left|R_0\right| - n/2\right| > c'\sqrt{n}$, then one of the sides has more than $n/2 + c'\sqrt{n}$ initial members, which we call the *majority side*. Now we apply Theorem 1.5 with ε replaced by $\varepsilon/2$. Notice that in the setting of Theorem 1.4 , if we have $p = \lambda n^{-1/2}$ for λ sufficiently large, then $c'\sqrt{n} \ge c/p$, where c is the constant in Theorem 1.5. Thus, by this theorem, the probability for the majority side to win is at least $1 - \varepsilon/2$, and we are done by the union bound. □

References

[1] Dan Alistarh, James Aspnes, David Eisenstat, Rati Gelashvili, and Ronald L. Rivest: Time-space trade-offs in population protocols. In *Proc. 28th Ann. ACM–SIAM Symp. on Discrete Algorithms (SODA'17)*, pp. 2560–2579. SIAM, 2017. [doi:10.1137/1.9781611974782.169, arXiv:1602.08032] 3

[2] Dan Alistarh, Rati Gelashvili, and Milan Vojnović: Fast and exact majority in population protocols. In *Proc. ACM Symp. Principles Distrib. Computing (PODC'15)*, pp. 47–56. ACM Press, 2015. [doi:10.1145/2767386.2767429] 3

[3] Itai Benjamini, Siu-On Chan, Ryan O'Donnell, Omer Tamuz, and Li-Yang Tan: Convergence, unanimity and disagreement in majority dynamics on unimodular graphs and random graphs. *Stoch. Proc. Appl.*, 126(9):2719–2733, 2016. [doi:10.1016/j.spa.2016.02.015] 3, 4

[4] Béla Bollobás: Models of random graphs. In *Random Graphs*, chapter 2, pp. 34–50. Cambridge Univ. Press, 2001. [doi:10.1017/CBO9780511814068.004] 2

[5] Peter Clifford and Aidan Sudbury: A model for spatial conflict. *Biometrika*, 60(3):581–588, 1973. [doi:10.1093/biomet/60.3.581] 3

[6] Morris H. DeGroot: Reaching a consensus. *J. Amer. Statistical Assoc.*, 69(345):118–121, 1974. [doi:10.2307/2285509] 3

[7] Carl-Gustav Esseen: A moment inequality with an application to the central limit theorem. *Scandinavian Actuarial J.*, 1956(2):160–170, 1956. [doi:10.1080/03461238.1956.10414946] 9

[8] Nikolaos Fountoulakis, Mihyun Kang, and Tamás Makai: Resolution of a conjecture on majority dynamics: Rapid stabilisation in dense random graphs. *Random Struct. Algor.*, 57(4):1134–1156, 2020. [doi:10.1002/rsa.20970, arXiv:1910.05820] 3, 4, 18

[9] WASSILY HOEFFDING: Probability inequalities for sums of bounded random variables. *J. Amer. Statistical Assoc.*, 58(301):13–30, 1963. [doi:10.1080/01621459.1963.10500830] 14

[10] SVANTE JANSON, TOMASZ ŁUCZAK, AND ANDRZEJ RUCINSKI: *Random Graphs*. Wiley, 2011. Chapter 1 (Preliminaries), pages 1–23. [doi:10.1002/9781118032718.ch1] 2

[11] JIŘI MATOUŠEK AND JAROSLAV NEŠETŘIL: *Invitation to Discrete Mathematics*. Oxford Univ. Press, 1998. LINK. 14

[12] ELCHANAN MOSSEL, JOE NEEMAN, AND OMER TAMUZ: Majority dynamics and aggregation of information in social networks. *Autonomous Agents and Multi-Agent Systems*, 28(3):408–429, 2014. [doi:10.1007/s10458-013-9230-4] 3, 4

[13] ELCHANAN MOSSEL AND OMER TAMUZ: Opinion exchange dynamics. *Probability Surveys*, 14:155–204, 2017. [doi:10.1214/14-PS230] 2, 3, 4

[14] ASHWIN SAH AND MEHTAAB SAWHNEY: Majority dynamics: The power of one, 2021. [arXiv:2105.13301] 6, 8

[15] IRINA GENNAD'EVNA SHEVTSOVA: An improvement of convergence rate estimates in the Lyapunov theorem. *Doklady Math.*, 82(3):862–864, 2010. [doi:10.1134/S1064562410060062] 9

[16] LINH TRAN AND VAN VU: Reaching a consensus on random networks: The power of few. In *Proc. 24th Internat. Conf. on Randomization and Computation (RANDOM'20)*, pp. 20:1–15. Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik, 2020. [doi:10.4230/LIPIcs.APPROX/RANDOM.2020.20] 1

[17] LINH TRAN AND VAN VU: The “power of few” phenomenon: The sparse case, 2023. [arXiv:2302.05605] 3, 4, 18

[18] ROMAN VERSHYNIN: Concentration of sums of independent random variables. In *High-Dimensional Probability: An Introduction with Applications in Data Science*, chapter 2, pp. 11–37. Cambridge Univ. Press, 2018. [doi:10.1017/9781108231596.005] 14

AUTHORS

Linh Tran
Graduate Student
Department of Mathematics
Yale University
New Haven, CT, USA
l.tran@yale.edu
https://math.yale.edu/people/linh-tran

Van Vu
Percey F. Smith Professor of Mathematics and Professor of Statistics & Data Science
Yale University
New Haven, CT, USA
van.vu@yale.edu
https://campuspress.yale.edu/vanvu/

ABOUT THE AUTHORS

LINH TRAN is currently a fifth year Ph. D. student in the Math department at Yale University, working under the supervision of Professor Van Vu. He went to special math classes for gifted children at the high school division of the Vietnam National University in Ho Chi Minh City. Encouraged and nurtured by his high school teachers, Tran found his passion for Math and was determined to follow this path, especially after having had little success in other subjects. Under the guidance of his advisor, Prof. Van Vu, he does research in random graph theory, particularly majority dynamics and generalizations, and random matrix theory, particularly its application in data analysis and machine learning methods. He is also interested in combinatorics problems involving the probabilistic method, applications of geometry in combinatorics, e. g., random fractals, and theoretical computer science.

VAN VU is the P. F. Smith Professor of Mathematics and Professor of Data Science at Yale University. Vu was born in Hanoi (Vietnam) in 1970. He went to special math classes for gifted children at Chu Van An and Hanoi-Amsterdam high schools. In 1987, he went to Hungary for his undergraduate studies, and obtained his M. Sc. in mathematics at Eötvös University, Budapest, in 1994. He received his Ph. D. at Yale University, in 1998 under the direction of László Lovász. His research interest includes random structures (random matrices, random graphs, random functions), probabilistic and additive combinatorics, and randomized algorithms. Recently, he he has also been interested in real-life applications of data science (such as NLP and computer vision).