THEORY OF COMPUTING, Volume 20 (4), 2024, pp. 1–13
www.theoryofcomputing.org

SPECIAL ISSUE: ANALYSIS OF BOOLEAN FUNCTIONS

Influential Coalitions for Boolean Functions I: Constructions

Jean Bourgain* Jeff Kahn† Gil Kalai‡

Received February 16, 2013; Revised October 12, 2020; Published September 30, 2024

Abstract. For $f : \{0,1\}^n \to \{0,1\}$ and $S \subset \{1,2,,\ldots,,n\}$, let $J_S^+(f)$ be the probability that, for x uniform from $\{0,1\}^n$, there is some $y \in \{0,1\}^n$ with $f(y) = 1$ and $x \equiv y$ outside S. We are interested in estimating, for given $\mu(f)$ ($:= \mathbb{E}(f)$) and m, the least possible value of $\max\{J_S^+(f) : |S| = m\}$.

A theorem of Kahn, Kalai, and Linial (KKL) gave some understanding of this issue and led to several stronger conjectures. Here we disprove a pair of conjectures from the late 80s, as follows.

(1) The KKL Theorem implies that there is a fixed $\alpha > 0$ so that if $\mu(f) \approx 1/2$, and $c > 0$, then there is a set S of size at most αcn with $J_S^+(f) \geq 1 - n^{-c}$. We show that for every $\delta > 0$ there is an f with $\mu(f) \approx 1/2$ and $J_S^+(f) \leq 1 - n^{-C}$ for every S of size $(1/2 - \delta)n$, where $C = C_\delta$. This disproves a conjecture of Benny Chor from 1989.

(2) We also show that for fixed $\delta > 0$ there are $c, \alpha > 0$ and Boolean functions f such that $\mu(f) > \exp[-n^{1-c}]$ and $J_S^+(f) \leq \exp[-n^{\alpha}]$ for each S of size $(1/2 - \delta)n$. This disproves a conjecture of the third author from the late 80s.

*Supported in part by NSF Grant DMS 1301619.
†Supported by NSF grant DMS0701175 and BSF grant 2006066.
‡Supported by BSF grant 2006066, ERC advanced grants 320924 and 834735, and NSF grant DMS-1300120.

ACM Classification: G.2.1, G.2.m

AMS Classification: 05D05, 68R05, 60C05

Key words and phrases: Boolean functions, influence, trace of set-systems, discrete isoperimetric inequalities, tribes

DOI: 10.4086/toc2024.v020a004

1 Introduction

For a set T we use $\Omega(T)$ for the discrete cube $\{0,1\}^T$ and μ_T for the uniform probability measure on $\Omega(T)$. In this paper f will always be a Boolean function on $\Omega([n])$ (that is, $f : \Omega([n]) \to \{0,1\}$, where, as usual, $[n] = \{1,\dots,n\}$). We write μ for $\mu_{[n]}$ and $\mu(f) = \mu(\{x : f(x) = 1\})$. We reserve x, y for elements of $\Omega([n])$ and set $|x| = \sum x_i$.

Following Ben-Or and Linial [3] we define, for a given f and $S \subset [n]$, the *influence of S toward one* to be

$$I_S^+(f) = \mu_{[n]\setminus S}(\{u \in \Omega([n]\setminus S) : \exists v \in \Omega(S), f(u,v) = 1\}) - \mu(f). \tag{1.1}$$

Similarly, the influence of S toward zero is

$$I_S^-(f) = \mu_{[n]\setminus S}(\{u \in \Omega([n]\setminus S) : \exists v \in \Omega(S), f(u,v) = 0\}) - (1-\mu(f)) \tag{1.2}$$

and the (total) influence of S is

$$I_S(f) = I_S^+(f) + I_S^-(f).$$

Suppose $\mu(f) = 1/2$. It then follows from a theorem of Kahn, Kalai and Linial [12] (Theorem 2.2 below, henceforth "KKL") that for every $a \in (0,1)$ there is an $S \subset [n]$ of size an with $I_S^+(f) \geq 1/2 - n^{-c}$, where $c > 0$ depends on a. (See Theorem 2.3.) Benny Chor conjectured in 1989 [7] that one can in fact achieve $I_S^+(f) \geq 1/2 - c^n$ (where, again, $c < 1$ depends on a). The conjecture has been "in the air" since that time, though as far as we know it has appeared in print only in [13, 14].

In this paper we disprove Chor's conjecture and another, similar conjecture from the same period. In the subsequent part II, we improve the preceding consequence of KKL.

For our purposes the subtracted terms in equations (1.1) and (1.2) are mostly a distraction, and it sometimes seems clearer to speak of $J_S^+(f) := I_S^+(f) + \mu(f)$ and $J_S^-(f) := I_S^-(f) + (1-\mu(f))$. Thus, for example, $J_S^+(f)$ is the probability that a uniform setting of the variables in $[n]\setminus S$ doesn't force $f = 0$, and Chor's conjecture predicts an S with $J_S^+(f) \geq 1 - c^n$. The following statement shows that this need not be the case.

Theorem 1.1. For any fixed $\alpha, \delta \in (0,1)$ there exists $C \in \mathbb{R}$ such that for all large enough n there exists an f such that $\mu(f) = \alpha$ and $J_S^+(f) < 1 - n^{-C}$ for every $S \subseteq [n]$ of size at most $(1/2-\delta)n$.

We should note that one cannot expect to go much beyond $|S| = (1/2-\delta)n$; for example if $\mu(f) = 1/2$, then it follows from the "Sauer–Shelah Theorem" (Theorem 2.5) that there is an S of size $n/2$ with $J_S^+(f) = 1$.

Another consequence of KKL (see Theorem 2.4 below) is that there is a $\beta > 0$ such that for any f with $\mu(f) > n^{-\beta}$ there is an S of size (say) $0.1n$ with influence $1 - o(1)$. A conjecture of the third author, again from the late 80s, asserts that the same conclusion holds even assuming only $\mu(f) > (1-\varepsilon)^n$ for sufficiently small ε. This conjecture turns out to be false as well.

Theorem 1.2. For any fixed $\varepsilon, \delta > 0$ there exists $\alpha > 0$ such that for all large enough n there exists an f such that $\mu(f) > (1-\varepsilon)^n$ and no set of size at most $(1/2-\delta)n$ has influence toward 1 more than $\exp[-n^\alpha]$.

This can be strengthened a bit to require $\mu(f) > \exp[-n^{1-c}]$ for some fixed $c = c_\delta > 0$.

The examples proving Theorems 1.1 and 1.2 are given in Section 3. Each of these is of the form $f = \bigwedge_{i=1}^{m} C_i$, where the C_i are random disjunctions of k literals using k distinct variables (henceforth "k-clauses"). These f's, which may be thought of as variants of the "tribes" construction of Ben-Or and Linial (see below), were inspired by a paper of Ajtai and Linial [1] and share with it the following curious feature. It is easy to see that any f can be converted to a *monotone* (i. e., increasing) f' with $\mathbb{E}(f') = \mu(f)$ and each influence (I_S^+ and so on) for f' no larger than the corresponding influence for f; thus it is natural to look for f's with small influences among the increasing functions. But the present random examples, like those of [1], do not do this, and it is not easy to see what one gets by monotonizing them.

Improving the consequences of KKL

While the preceding, rather optimistic conjectures turn out to be false, in part II we show that the first of the aforementioned consequences of KKL can be improved. (The results and proofs of both parts were given together in the arXiv publication [6].)

Theorem 1.3. For each $C > 0$, there is a $\delta > 0$ such that for any f with $\mu(f) > n^{-C}$, there is an $S \subset [n]$ of size at most $(1/2 - \delta)n$ with $J_S^+(f) > .9$.

(Of course, as elsewhere in this discussion, ".9" could be any preset $\rho < 1$.)

Note that the gap between Theorems 1.2 and 1.3 is substantial and our modest progress is likely not the final word on the problem. For example, could it be that there is some fixed β such that there are f's with $\mu(f) > n^{-\beta}$ for which no S of size $0.1n$ has influence $\Omega(1)$? We will discuss this question further in the next section.

The Proof of Theorem 1.3 ([6]) goes roughly as follows. We employ two strategies, both variants of the analysis in [12]. The first uses the total influence, assumed sufficiently large. If at some point this total influence becomes "small," we switch to a different procedure that combines the incremental argument from [12] with the Sauer–Shelah theorem.

2 Background and perspective

Influence

We write $I_\ell(f)$ for $I_{\{\ell\}}(f)$. A form of the classic edge isoperimetric inequality for Boolean functions is

Theorem 2.1. For any (Boolean function) f with $\mu(f) = t$,

$$I(f) := \sum_{k=1}^{n} I_\ell(f) \geq 2t \log_2(1/t). \tag{2.1}$$

(This convenient version is easily derived from the precise statement, due to Hart [9]; see also [11, Sec. 7] for a simple inductive proof.)

While (2.1) is exact or close to exact (depending on t), it typically gives only a weak lower bound on the maximum of the $I_\ell(f)$'s, namely

$$\max_\ell I_\ell(f) \geq 2t \log_2(1/t)/n. \tag{2.2}$$

For t not too close to 0 or 1, the following statement from [12] gives better information.

Theorem 2.2 (KKL)**.** There is a fixed $c > 0$ such that for any f with $\mu(f) = t$, there is an $\ell \in [n]$ with

$$I_\ell(f) \geq ct(1-t)\log n/n. \tag{2.3}$$

Recall that $J_\ell(f) = \mu(f) + I_\ell(f)$. Repeated application of Theorem 2.2 gives the following two corollaries.

Theorem 2.3. For all $a, t \in (0,1)$ there is a c such that for any f with $\mu(f) = t$ there is an $S \subseteq [n]$ with $|S| \leq an$ and

$$J_S^+(f) \geq 1 - n^{-c}$$

(that is, $I_S^+(f) \geq (1-t) - n^{-c}$).

Similarly (either by the same argument or by applying Theorem 2.3 to the function $1 - f(x)$) there is a small S' with $J_{S'}^-(f) \geq 1 - n^{-c}$ (i. e., $I_{S'}^-(f) \geq t - n^{-c}$), and combining these observations we find that there is in fact a small S'' (e. g., $S \cup S'$) with $I_{S''}(f) \geq 1 - n^{-c}$.

Theorem 2.4. For every $\delta, \epsilon > 0$, there is an $\alpha > 0$ such that for large enough n and any f with $\mu(f) \geq n^{-\alpha}$, there is an $S \subseteq [n]$ with $|S| = \delta n$ and

$$J_S^+(f) \geq 1 - \epsilon.$$

The conjecture of Chor stated in Section 1 asserts that the n^{-c} in Theorem 2.3 can be replaced by something exponential in n, and the conjecture stated before Theorem 1.2 proposes a similar weakening of the $n^{-\alpha}$ lower bound on $\mu(f)$ in Theorem 2.4. As already noted, we will show below that these conjectures are incorrect.

Tribes

The original "tribes" examples of Ben-Or and Linial [3] are Boolean functions of the form $f = \bigvee_{i=1}^m C_i$, where the "tribes" C_i are conjunctions of k (distinct) variables and each variable belongs to exactly one tribe. The dual of such an f (so "dual tribes") is $g = \bigwedge_{i=1}^m D_i$, where D_i is the disjunction of the variables in C_i (so again, each variable belongs to exactly one D_i).

When $k = \log n - \log\log n - \log\ln(1/t)$, we have $1 - \mu(f) = \mathbb{E}(g) \approx t$ (where $\log = \log_2$ and f, g are as above). For fixed $t \in (0,1)$ both constructions show that Theorem 2.2 is sharp (up to the value of c).

On the other hand, when $t = O(n^{-c})$ for a fixed $c > 0$, f shows that (2.2) is tight up to a multiplicative constant, depending on c; for example, $k = 2\log n - \log\log n$ gives $\mu(f) \approx 1/(2n)$

and $I_\ell(f) \approx 2\log n/n^2 = \Theta(\mu(f)\log(1/\mu(f))/n)$ for each ℓ. (In contrast, for $\mathbb{E}(g) \approx 1/n$, we should take $k = \log n - 2\log\log n - 1$, in which case $I_\ell(g) = \Theta(\log^2 n/n^2)$ and (2.2) is off by a log.)

For f (again, as above) with $\mu(f) \in (\Omega(1), 1-\Omega(1))$, there are sets of size $\log n$ with large influence toward 1, while no set of size $o(n/\log n)$ has influence $\Omega(1)$ toward 0. (The corresponding statement with the roles of 0 and 1 reversed holds for g.) The Ajtai-Linial construction mentioned in the introduction shows that there are Boolean functions h with $\mathbb{E}(h) \approx 1/2$ and $I_S(h) < o(1)$ for every S of size $o(n/\log^2 n)$.

Trace

We now briefly consider influences from a different point of view. For a set X let $2^X = \{S : S \subset X\}$, $\binom{X}{k} = \{S \subset X : |S| = k\}$, $\binom{X}{<k} = \{S \subset X : |S| < k\}$ and $\binom{n}{<k} = \sum_{i=0}^{k-1}\binom{n}{i}$. For $\mathcal{F} \subset 2^X$ and $Y \subset X$, the *trace* of $\mathcal{F}$ on Y is

$$\mathcal{F}_{|Y} = \{S \cap Y : S \in \mathcal{F}\}.$$

Let $X = [n]$. The "Sauer–Shelah Theorem" (below) determines, for every n and m, the minimum T such that for each $\mathcal{F} \subseteq 2^X$ of size T there is some $Y \in \binom{X}{m}$ on which the trace of $\mathcal{F}$ is *complete*, meaning $\mathcal{F}_{|Y} = 2^Y$. Such a Y is said to be *shattered* by $\mathcal{F}$.

Theorem 2.5 (The Sauer–Shelah Theorem)**.** If $\mathcal{F} \subset 2^X$ and $|\mathcal{F}| > \binom{n}{<r}$, then $\mathcal{F}$ shatters some $Y \in \binom{X}{r}$.

That this is sharp is shown by $\mathcal{F} = \binom{X}{<r}$, the *Hamming ball* of radius $r-1$ about $\emptyset$ with respect to the usual Hamming metric on $2^X \equiv \Omega(X)$.

Theorem 2.5 was proved around the same time by Sauer [15], Shelah and Perles [16], and Vapnik and Chervonenkis [17]. It has many connections, applications and extensions in combinatorics, probability theory, model theory, analysis, statistics and other areas.

We identify $\Omega([n])$ and $2^{[n]}$ in the usual way. The connection between traces and influences is as follows. Let f be a Boolean function on $\Omega([n])$ and $\mathcal{F} = f^{-1}(1)$. It is easy to see that for $S \subseteq [n]$ and $T = [n]\backslash S$,

$$J_S^+(f) = 2^{-|T|}|\mathcal{F}_{|T}|.$$

Thus, in the language of traces, we are interested in the effect of relaxing "$\mathcal{F}$ shatters Y" to require only that $\mathcal{F}_{|Y}$ contain a large fraction of 2^Y.

The following arrow notation (e. g., [5, 8, 2]) is convenient. Write $(N, n) \to (M, r)$ if every $\mathcal{F} \subseteq 2^{[n]}$ of size N has a trace of size at least M on some $S \in \binom{[n]}{r}$; for example the Sauer–Shelah Theorem says $(\binom{n}{<r}+1, n) \to (2^r, r)$.

One might hope that Hamming balls would again give the best examples in our relaxed setting, which would say, for example, that for $m \le n$,

$$(\tbinom{n}{<r}+1, n) \to (\tbinom{m}{<r}+1, m). \tag{2.4}$$

But (2.4), which was first considered by Bollobás and Radcliffe [4] and would have implied both of the conjectures disproved here, was shown in [4] to be false for fixed r and (large) $m = n/2$. (For $r = n/2$ and $m = n-1$, it fails for the original tribes example discussed above.)

Two problems

Question 2.1. For fixed $\alpha, \delta > 0$, what is the largest $t \in (0, 1/2)$ for which one can find Boolean functions f with $\mu(f) = t$ and $I_S^+(f) < \alpha$ for every $S \subseteq [n]$ of size at most $(1/2 - \delta)n$?

As far as we know $t > n^{-\beta}$ (with β depending on α, δ) is possible. The influence of sets of half the variables is of special interest:

Question 2.2. Given $\mu(f) = o(1)$ what can be said about the maximum of the $J_S^+(f)$'s for $|S| = n/2$? What is the smallest t such that for each f with $\mu(f) = t$ there is some S of size $n/2$ with $J_S^+(f) \geq 1/2$? (Here we assume that n is even.)

3 Boolean functions without influential coalitions

In each construction we consider, for suitable k and m, $f = \bigwedge_{i=1}^m C_i$, where the C_i are random disjunctions of k literals using k distinct variables (henceforth "k-clauses") and show that f is likely to have the desired properties. Every C_i can be regarded as a list of specifications for the values of k variables. We use g_i for the specification associated with C_i, and write $C_i \sim x$ if some entry of x agrees with g_i. We say C_i *misses* $S \subseteq [n]$ if the indices of all variables in C_i lie in $[n] \setminus S$.

Let $s = (1/2 - \delta)n$. We will always use S for an s-subset of $[n]$ and (for such an S) set $m_S = |\{i : C_i \text{ misses } S\}|$. (Following common practice we omit irrelevant floor and ceiling symbols, pretending all large numbers are integers. As in the case of k, m and s, parameters not declared to be constants are assumed to be functions of n.) We use log for $\log_2$.

Both constructions will make use of the next two observations, with Theorem 1.1 following immediately from these and Theorem 1.2 requiring a little more work.

Lemma 3.1. *If $k = o(\sqrt{n})$ and $(1/2 + \delta)^k m = \omega(n)$ then w. h. p.*

$$m_S \sim (1/2 + \delta)^k m \quad \textit{for all } S \in \binom{[n]}{s}. \tag{3.1}$$

(where, as usual, $a_n \sim b_n$ means $a_n / b_n \to 1$ and *with high probability* (w. h. p.) means with probability tending to 1, both as $n \to \infty$).

Proof. For a given S, m_S has the binomial distribution $B(m, p)$, with $p = \binom{n-s}{k} / \binom{n}{k} \sim (1/2 + \delta)^k$ (using $k = o(\sqrt{n})$ for the "$\sim$"). Thus $\mathbb{E} m_S = mp$ and, by "Chernoff's Inequality"* (e. g., [10, Theorem 2.1]),

$$\Pr(m_S \notin ((1 - \zeta)mp, (1 + \zeta)mp)) < \exp[-\Omega(\zeta^2 mp)], \tag{3.2}$$

for $\zeta \in (0, 1)$. Applying this with a ζ which is both $\omega(\sqrt{n/(mp)})$ and $o(1)$ gives $\Pr(m_S \nsim mp) < 2^{-\omega(n)}$, and the union bound then gives (3.1). □

The next lemma is stated to cover both applications, though nothing so precise is needed for Theorem 1.1.

*Though now commonly called a "Chernoff bound," this and related inequalities essentially go back to Sergei Natanovich Bernstein in the early part of the 20th century; see [18].

Lemma 3.2. *If there is a* ξ *for which*

$$\exp[-\xi^2 n] = o((1-2^{-k})^m) \tag{3.3}$$

and

$$[(1+2\xi)/4]^k = o(1/m), \tag{3.4}$$

then w. h. p.

$$\mu(f) \sim (1-2^{-k})^m. \tag{3.5}$$

Proof. This is a simple second moment method calculation (similar to what is done in [1], though described differently there).

Recalling that x, y always denote elements of $\{0,1\}^n$, write A_x for the event $\{f(x)=1\}$ and $\mathbf{1}_x$ for its indicator, and set $X = \sum \mathbf{1}_x = 2^n \mu(f)$. Then $\Pr(A_x) = (1-2^{-k})^m$ and $\mathbb{E}X = (1-2^{-k})^m 2^n$; so we just need to show $\mathbb{E}X^2 \sim \mathbb{E}^2 X$ (equivalently, $\mathbb{E}X^2 < (1+o(1))\mathbb{E}^2X$), since Chebyshev's Inequality then gives $\Pr(|X-\mathbb{E}X| > \zeta\mathbb{E}X) = o(1)$ for any fixed $\zeta > 0$.

We have

$$\mathbb{E}X^2 = \sum_x \sum_y \mathbb{E}\mathbf{1}_x\mathbf{1}_y = \sum_x \Pr(A_x) \sum_y \Pr(A_y|A_x),$$

so will be done if we show that for a fixed x,

$$\sum_y \Pr(A_y|A_x) < (1+o(1))(1-2^{-k})^m 2^n.$$

Since the sum is the same for all x, it is enough to prove this when $x = \underline{0}$. Set $Z = \{y : |y| < (1/2-\xi)n\}$ and recall that by Chernoff's Inequality (3.2), $|Z| < \exp[-2\xi^2 n]2^n$. It is thus enough to show that (for $x = \underline{0}$)

$$y \notin Z \;\Rightarrow\; \Pr(A_y|A_x) < (1+o(1))(1-2^{-k})^m, \tag{3.6}$$

since then, using (3.3), we have

$$\sum_y \Pr(A_y|A_x) \;<\; |Z| + \sum_{y\notin Z} \Pr(A_y|A_x) \;<\; (1+o(1))(1-2^{-k})^m 2^n.$$

Now since $x = \underline{0}$, we have $A_x = \{g_i \neq \underline{1} \quad \text{for all } i\}$; so if, for a given $y \notin Z$, we set $\beta = \beta_y = \Pr(C_i \sim y | g_i \neq \underline{1})$ (a function of $|y|$), then $\Pr(A_y|A_x) = \beta^m$. Aiming for a bound on β, we have

$$\begin{aligned} 1-2^{-k} &= \Pr(C_i \sim y) \\ &= \Pr(g_i = \underline{1})\Pr(C_i \sim y|g_i = \underline{1}) + \Pr(g_i \neq \underline{1})\Pr(C_i \sim y|g_i \neq \underline{1}) \\ &= 2^{-k}\Pr(C_i \sim y|g_i = \underline{1}) + (1-2^{-k})\beta \end{aligned}$$

and

$$\Pr(C_i \sim y|g_i = \underline{1}) > 1-(1-|y|/n)^k \geq 1-(1/2+\xi)^k =: 1-\nu$$

(using the fact that if $g_i = \underline{1}$, then $g_i \nsim y$ iff all indices of variables in C_i belong to $\{j : y_j = 0\}$). Combining, we have

$$\beta < (1-2^{-k})^{-1}[1-2^{-k}-2^{-k}(1-\nu)] = (1-2^{-k})\left[1+\tfrac{2^{-k}(\nu-2^{-k})}{(1-2^{-k})^2}\right],$$

which with (3.4) gives $\beta^m < (1+o(1))(1-2^{-k})^m$ (which is (3.6)). □

Proof of Theorem 1.1. Notice that it is enough to prove this with $\mu(f) \sim \alpha$ (rather than "$= \alpha$"); for then, since $f^{-1}(1) \subseteq g^{-1}(1)$ trivially implies $J_S^+(f) \le J_S^+(g)$ for all S, we can choose $\beta \in (\alpha, 1)$ and a g with $\mathbb{E}(g) \sim \beta$ possessing the desired small influences, and shrink $g^{-1}(1)$ to produce f.

Let $k = C \log n$, with $C = C_\delta$ chosen so that $(1+2\delta)^k = \omega(n)$ (e. g., $C = 1/\delta$ does this), and $m = 2^k \ln(1/\alpha) = n^C \ln(1/\alpha)$. Here all we use from Lemma 3.1 (whose hypotheses are satisfied for our choice of k and m) is the fact that w. h. p. $m_S \neq 0$ for all S, whence each $J_S^+(f)$ is at most $1-2^{-k} = 1-n^{-C}$. On the other hand, by Lemma 3.2 (with, for example, $\xi = 0.1$), we have $\mu(f) \sim \alpha$ w. h. p. So w. h. p. f meets our requirements. □

Proof of Theorem 1.2. . Here, intending to recycle n, m and f, we rename these quantities $\mathbf{n}$, $\mathbf{m}$ and $\mathbf{f}$. We may of course assume δ is fairly small. Let (for example) $\xi = \delta/3$, fix ε with $0 < \varepsilon < \xi^2$, and set $k = (1+\delta)\log \mathbf{n}$ and $\mathbf{m} = \varepsilon 2^k \mathbf{n}$. These values are easily seen to give the hypotheses of Lemmas 3.1 and 3.2. In particular, we can say that w. h. p. the supports of the C_i are chosen so (3.1) holds (note this says nothing about the values specified by the g_i) and

$$\mathbb{E}(\mathbf{f}) \sim (1-2^{-k})^{\mathbf{m}} \sim e^{-\varepsilon \mathbf{n}}. \tag{3.7}$$

Set $n = (1/2+\delta)\mathbf{n}$. Fix $S \in \binom{[\mathbf{n}]}{s}$, set $m = m_S$, and let $f = f_S$ be the conjunction of the m clauses C_i—w.l.o.g. $C_1, , \ldots, , C_m$—that miss S. Thus f is the conjunction of $m \sim (1/2+\delta)^k \mathbf{m} = \varepsilon(1+2\delta)^k \mathbf{n}$ random k-clauses from a universe of n variables. Theorem 1.2 (with $\alpha = \delta$) thus follows from

Claim A. $\Pr(\mu(f) > \exp[-n^\delta]) < o(2^{-\mathbf{n}})$

(since then w. h. p. we have $\mathbb{E}(f_S) \le \exp[-n^\delta]$ for every S).

Remarks. The actual bound in Claim A will be $\exp[-\Omega(m)]$, so much smaller than $2^{-\mathbf{n}}$. Note that here it does not matter whether we take μ to be our original measure (i. e., μ uniform on $\{0,1\}^{\mathbf{n}}$) or uniform measure on $Q := \{0,1\}^n$; but it is now more natural to think of the latter—and we will do so in what follows—since our original universe plays no further role in this discussion. It may also be worth noting that, unlike in the proof of Lemma 3.2, the second moment method is not strong enough to give the exponential bound in Claim A.

Claim B. If $X \subseteq Q$, $\mu(X) = \beta > \exp[-o(n/\log^2 n)]$ and $\zeta = o(2^{-k})$, then for a random k-clause C,

$$\Pr(\mu(C \wedge X) > (1-\zeta)\mu(X)) < 1/2$$

(where $C \wedge X = \{x \in X : C \sim x\}$).

Remark. This is probably true for β greater than something like $\exp[-n/k]$. The bound in the claim is just what the proof gives, and is more than enough for us since we are really interested in much larger β.

To see that Claim B implies Claim A, set $f_j = \bigwedge_{i=1}^{j} C_i$ and notice that $\mu(f) \geq \beta$ implies (for example)

$$\left|\left\{i : \mathbb{E}(f_i) < \left(1 - \tfrac{5\ln(1/\beta)}{m}\right)\mathbb{E}(f_{i-1})\right\}\right| < m/5 \tag{3.8}$$

(and, of course, $\mathbb{E}(f_i) \geq \beta$ for all i). But if we take $\beta = \exp[-n^\delta]$ then our choice of parameters gives

$$\zeta := 5m^{-1}\ln(1/\beta) = o(2^{-k})$$

(using $m/\ln(1/\beta) = \Theta(n^{(1+\delta)\log(1+2\delta)+1-\delta})$ and $2^k = n^{1+\delta}$), so Claim B bounds the probability of (3.8) by

$$\tbinom{m}{m/5}2^{-4m/5} = o(2^{-\mathbf{n}}). \qquad \square$$

Proof of Claim B. Let G be the bipartite graph on $Q \cup W$, where W is the set of $(n-k)$-dimensional subcubes of Q and, for $(x, D) \in Q \times W$, we take $x \sim D$ if $x \in D$. (So we have gone to complements: for a clause C the corresponding subcube is $D = \{y : C \nsim y\}$, so $C \wedge X = X \setminus D$.)

Assuming Claim B fails at X, fix $T \subseteq W$ with $|T| = |W|/2$ and

$$D \in T \;\Rightarrow\; \mu(D \cap X) < \zeta\mu(X),$$

and set $R = W \setminus T$.

Consider the experiment: (i) choose x uniformly from X; (ii) choose D uniformly from the members of W containing x; (iii) choose y uniformly from D.

Claim C. $\Pr(y \in X) > (2 - o(1))\beta$.

Proof. Since each triple (x, D, y) with $x \in X$ and $x, y \in D$ is produced by (i)-(iii) with probability $|X|^{-1} \cdot 2^k|W|^{-1} \cdot 2^{k-n}$, we just need to show that the number of such triples with $y \in X$ is at least

$$(2 - o(1))\beta|X||W|2^n 2^{-2k} = (2 - o(1))|X|^2|W|2^{-2k}.$$

Writing d for degree in G, we have

$$\sum_{x \in X} d_T(x) = \sum_{D \in T} d_X(D) < |T|\zeta|X|,$$

implying

$$\begin{aligned}\sum_{D \in R} d_X(D) &= \sum_{x \in X}(d(x) - d_T(x)) \\ &> |X||W|2^{-k} - \zeta|T||X| = (1 - o(1))|X||W|2^{-k}.\end{aligned}$$

The number of triples (x, D, y) as above is thus

$$\sum_{D\in W} d_X^2(D) \geq \sum_{D\in R} d_X^2(D) \geq \left(\sum_{D\in R} d_X(D)\right)^2 /|R|$$
$$> (1-o(1))|X|^2|W|^2|R|^{-1}2^{-2k} = (2-o(1))|X|^2|W|2^{-2k}\,. \qquad \square$$

Let $T(x)$ be the random element of Q obtained from x by choosing K uniformly from $\binom{[n]}{k}$ and randomly (uniformly, independently) resampling the x_i for $i \notin K$. Then y obtained from x by (ii) and (iii) above is just $T(x)$, so the next assertion contradicts Claim C, completing the proof of Claim B (and Theorem 1.2).

Claim D. If $\mu(X) > \exp[-o(n/\log^2 n)]$ and x is uniform from X, then $\Pr(T(x) \in X) < (1+o(1))\mu(X)$.

Remark. If X is a subcube of codimension n/k, say $X = \{x : x \equiv 0 \text{ on } L\}$ with $|L| = n/k$, then for any $x \in X$,

$$\Pr(T(x) \in X) = \sum_t \Pr(|K \cap L| = t)2^{-(|L|-t)} = \mu(X) \sum_t \Pr(|K \cap L| = t)2^t,$$

and, since $|K \cap L|$ is essentially Poisson with mean 1, the sum is approximately $\mathrm{e}^{-1}\sum_t 2^t/t! = \mathrm{e}$. So Claim D fails for $\mu(X) = 2^{-n/k}$ and, as earlier, it is natural to guess that it holds if $\mu(X)$ is much greater than this.

Proof of Claim D. Let $Q_r = \{y \in Q : |y| \leq r\}$. The assumption on $\mu(X)$ implies that $\mu(Q_{r-1}) < \mu(X) \leq \mu(Q_r)$ for some $r > (1/2 - o(1/k))n$, so Claim D follows from

Claim E. If $\varphi = o(1/k)$ and $r > (1/2 - \varphi)n$, then for any $x \in Q$ and $X \subseteq Q$ with

$$\mu(X) \leq \mu(Q_r), \tag{3.9}$$

we have

$$\Pr(T(x) \in X) < (1 + o(1))\mu(Q_r). \tag{3.10}$$

Proof. We may assume $x = \underline{0}$, so that $\Pr(T(x) = y)$ is a decreasing function of $|y|$. We thus maximize $\Pr(T(x) \in X)$ subject to (3.9) by taking $X = Q_r$, and (3.10) is then a routine calculation using

$$\mu(X) = \Pr(\mathrm{Bin}(n, 1/2) \leq r)$$

and

$$\Pr(T(x) \in X) = \Pr(\mathrm{Bin}(n-k, 1/2) \leq r)$$

(where $\mathrm{Bin}(\cdot,\cdot)$ denotes a binomially distributed r.v.). $\square$

Acknowledgment We would like to thank Roy Meshulam for helpful conversations. Part of this work was carried out while the second author was visiting Jerusalem under BSF grant 2006066. The first author thanks the UC Berkeley Mathematics Department for its hospitality.

References

[1] MIKLÓS AJTAI AND NATHAN LINIAL: The influence of large coalitions. *Combinatorica*, 13(2):129–145, 1993. [doi:10.1007/BF01303199] 3, 7

[2] NOGA ALON, GUY MOSHKOVITZ, AND NOAM SOLOMON: Traces of hypergraphs. *J. London Math. Soc.*, 100(2):498–517, 2019. [doi:10.1112/jlms.12233] 5

[3] MICHAEL BEN-OR AND NATHAN LINIAL: Collective coin flipping. In SILVIO MICALI, editor, *Randomness and Computation*, volume 5 of *Advances in Computing Research*, pp. 91–115. JAI Press, 1989. DSpace@MIT. Preliminary version in FOCS'85. 2, 4

[4] BÉLA BOLLOBÁS AND ANDREW J RADCLIFFE: Defect Sauer results. *J. Combin. Theory–A*, 72(2):189–208, 1995. [doi:10.1016/0097-3165(95)90060-8] 5

[5] JOHN ADRIAN BONDY: Induced subsets. *J. Combin. Theory–B*, 12(2):201–202, 1972. [doi:10.1016/0095-8956(72)90025-1] 5

[6] JEAN BOURGAIN, JEFF KAHN, AND GIL KALAI: Influential coalitions for Boolean functions, 2014. [arXiv:1409.3033] 3

[7] BENNY CHOR: Private communication, circa 1990. 2

[8] PÉTER FRANKL: On the trace of finite sets. *J. Combin. Theory–A*, 34(1):41–45, 1983. [doi:10.1016/0097-3165(83)90038-9] 5

[9] SERGIU HART: A note on the edges of the n-cube. *Discr. Math.*, 14(2):157–163, 1976. [doi:10.1016/0012-365X(76)90058-3] 3

[10] SVANTE JANSON, TOMASZ ŁUCZAK, AND ANDRZEJ RUCIŃSKI: *Random Graphs*. Wiley, 2000. [doi:10.1002/9781118032718] 6

[11] JEFF KAHN AND GIL KALAI: Thresholds and expectation thresholds. *Combin. Probab. Comput.*, 16(3):495–502, 2007. [doi:10.1017/S0963548307008474] 3

[12] JEFF KAHN, GIL KALAI, AND NATHAN LINIAL: The influence of variables on Boolean functions. In *Proc. 29th FOCS*, pp. 68–80. IEEE Comp. Soc., 1988. [doi:10.1109/SFCS.1988.21923] 2, 3, 4

[13] GIL KALAI: Nati's influence. *Combinatorics and More, Gil Kalai's blog*, May 2008. LINK. 2

[14] NATHAN LINIAL: Open problem. In JEFF KAHN, ANGELIKA STEGER, AND BENNY SUDAKOV, editors, *Oberwolfach Report No. 01/2011*, pp. 75–76. 2011. LINK. 2

[15] NORBERT SAUER: On the density of families of sets. *J. Combin. Theory–A*, 13(1):145–147, 1972. [doi:10.1016/0097-3165(72)90019-2] 5

[16] SAHARON SHELAH: A combinatorial problem; stability and order for models and theories in infinitary languages. *Pacific J. Math*, 41(1):247–261, 1972. Project Euclid. 5

[17] VLADIMIR N. VAPNIK AND ALEXEI YA. CHERVONENKIS: On the uniform convergence of relative frequencies of events to their probabilities. *Theory of Probability & Its Applications*, 16(2):264–280, 1971. Reproduced in Measure of Complexity: Festschrift for Alexei Chervonenkis, Springer 2015. [doi:10.1137/1116025] 5

[18] WIKIPEDIA: Bernstein inequalities (probability theory). Accessed 06-09-2024. 6

AUTHORS

Jean Bourgain
Past Professor
Department of Mathematics
Institute for Advanced Study
Princeton NJ 08540
https://www.ias.edu/scholars/bourgain

Jeff Kahn
Professor
Department of Mathematics
Rutgers University
Piscataway NJ 08854
jkahn@math.rutgers.edu
https://sites.math.rutgers.edu/~jkahn/

Gil Kalai
Professor
Einstein Institute of Mathematics
Hebrew University of Jerusalem
Jerusalem, Israel
and
Efi Arazi School of Computer Science
Reichman University, Herzliya.
kalai@math.huji.ac.il
http://www.ma.huji.ac.il/~kalai/

ABOUT THE AUTHORS

JEAN BOURGAIN (1954–2018) received his Ph. D. from the Vrije Universiteit Brussel in 1977. He was professor at I.H.E.S., the University of Illinois, and from 1994, at the Institute for Advanced Study in Princeton. He worked on functional analysis, harmonic analysis, ergodic theory, partial differential equations, number theory, and combinatorics. He was a recipient of the Fields Medal (1994), the Shaw Prize (2010), and the Breakthrough Prize (2017).

JEFF KAHN received his Ph. D. from The Ohio State University in 1979, under the direction of D. K. Ray-Chaudhuri, and, following some time at MIT, has been at Rutgers since 1984. He works on discrete mathematics and related topics, and is a recipient of the Pólya Prize (1996) and the Fulkerson Prize (2012).

GIL KALAI graduated from the Hebrew University of Jerusalem in 1983; his advisor was Micha A. Perles. He works on geometric and probabilistic combinatorics, convexity, and linear programming. He is known for his blog, his interest in advancing women in academia, and his "argument against quantum computers."